Stranger things – the fractal monsters that lurk between dimensions

Richard Smith

18th October 2017

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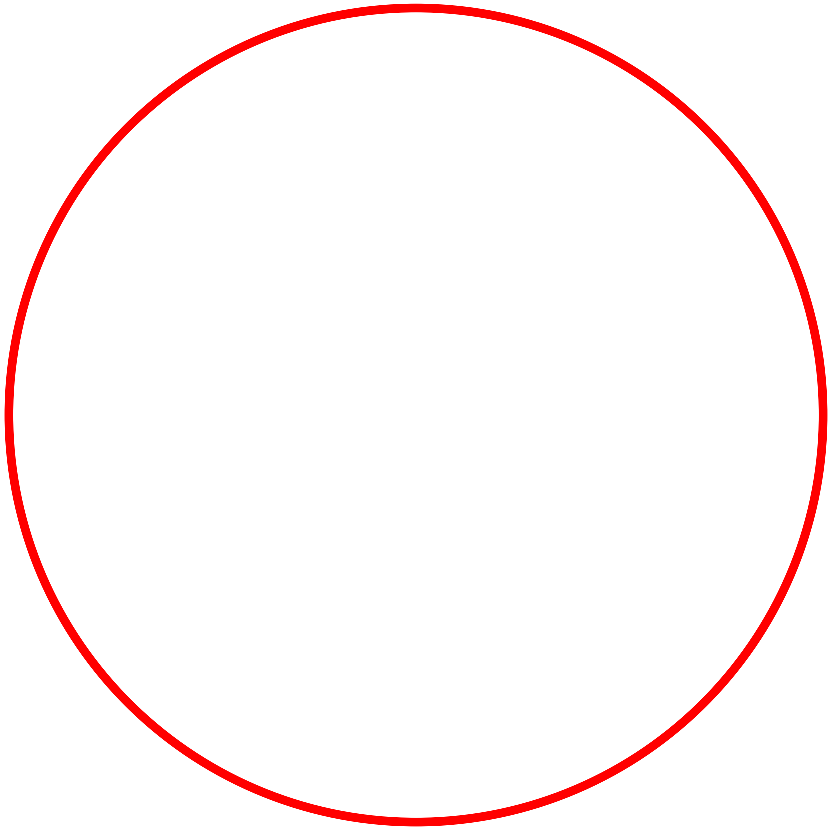
In school we learned about things from geometry such as. . .



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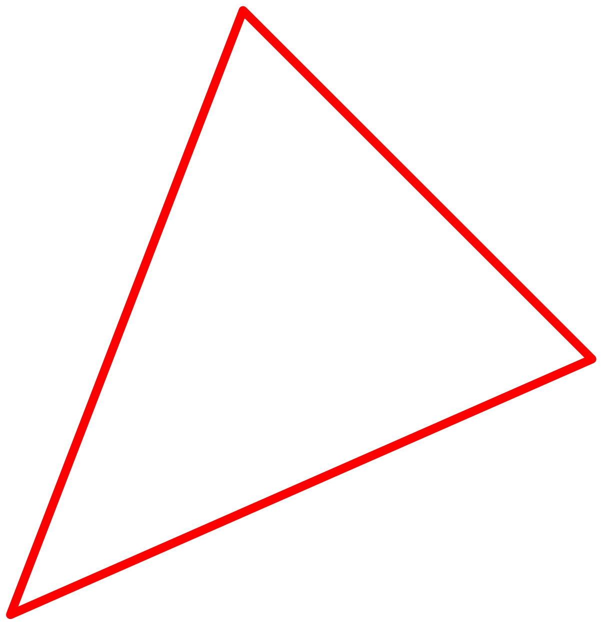

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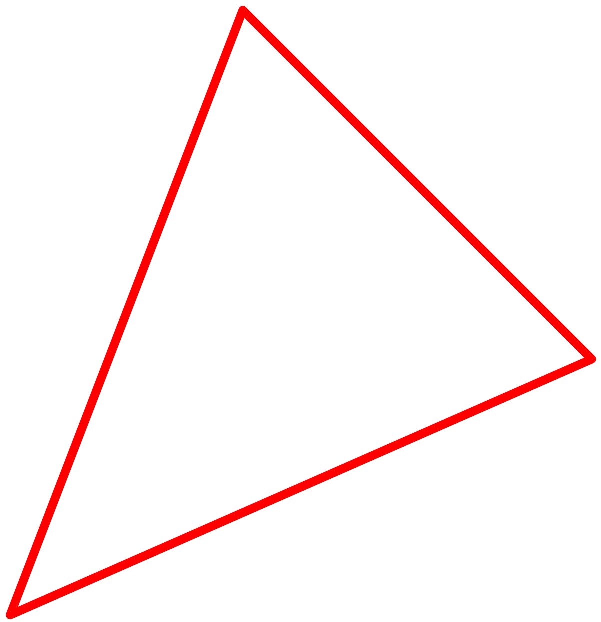

These things are nice and (mainly) smooth and cuddly and **regular**.

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$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3}{4}x^2 - \frac{1}{2}x + \frac{1}{2} \qquad \text{and equals } \frac{1}{2} \text{ at } x = 0.$$

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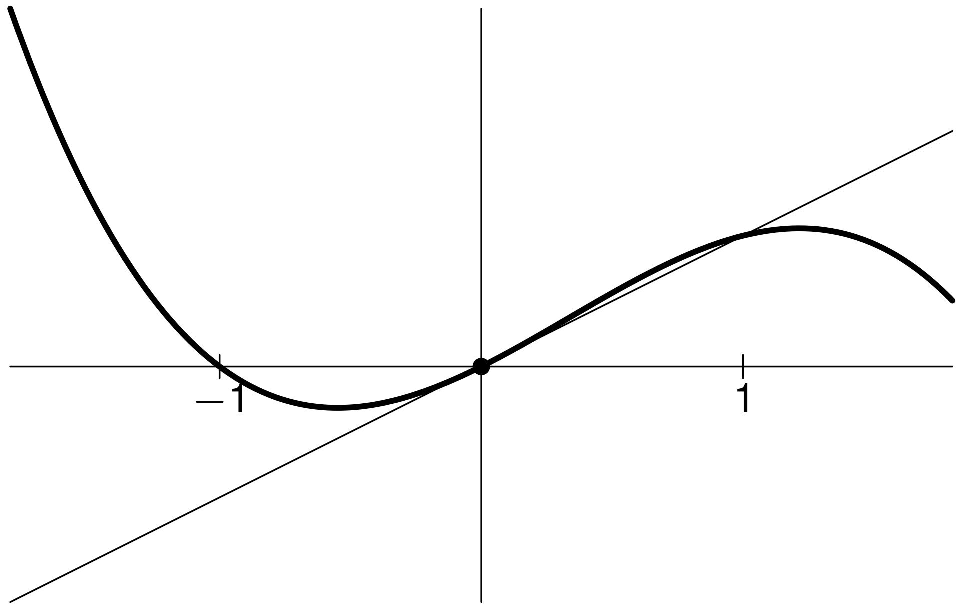

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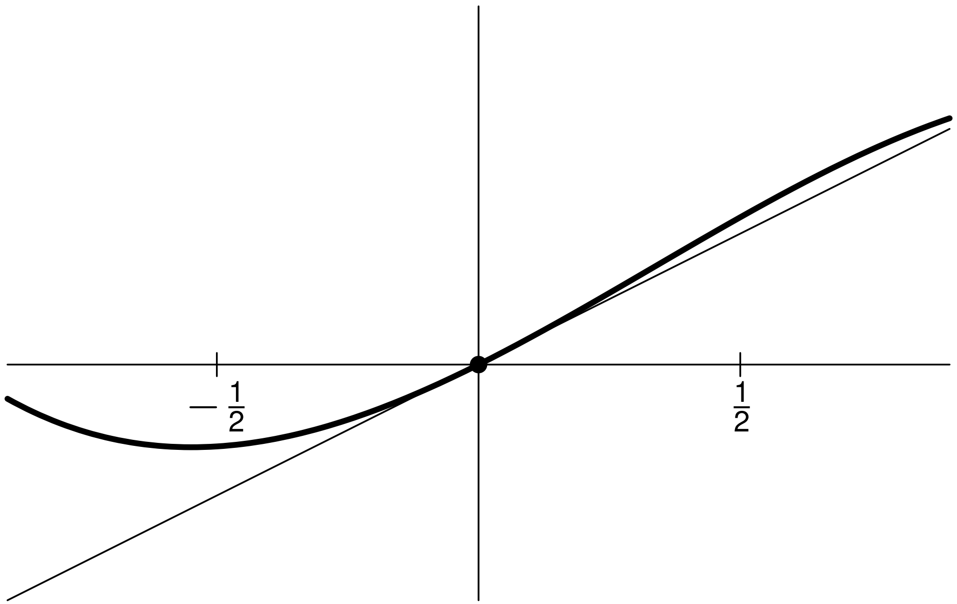

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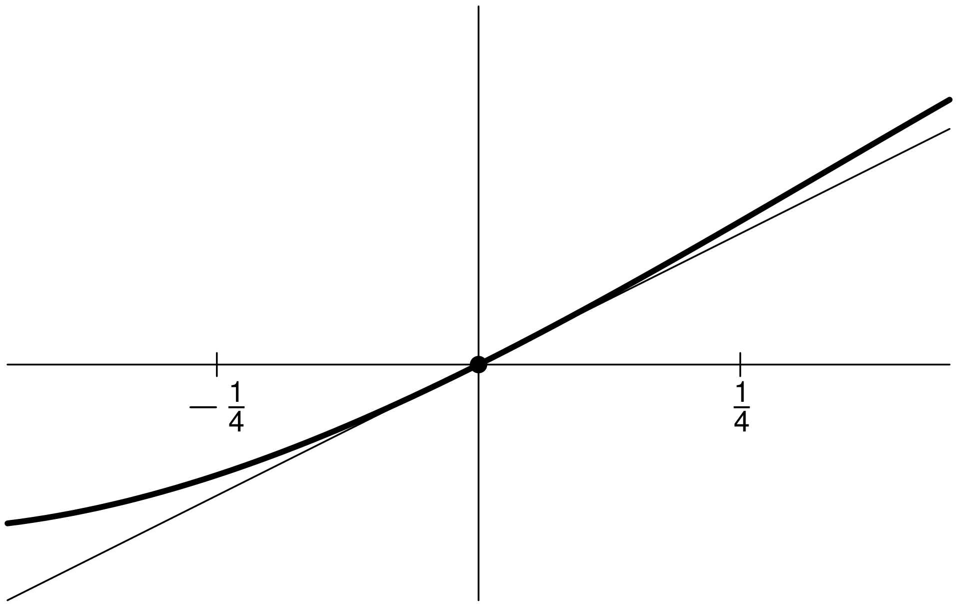

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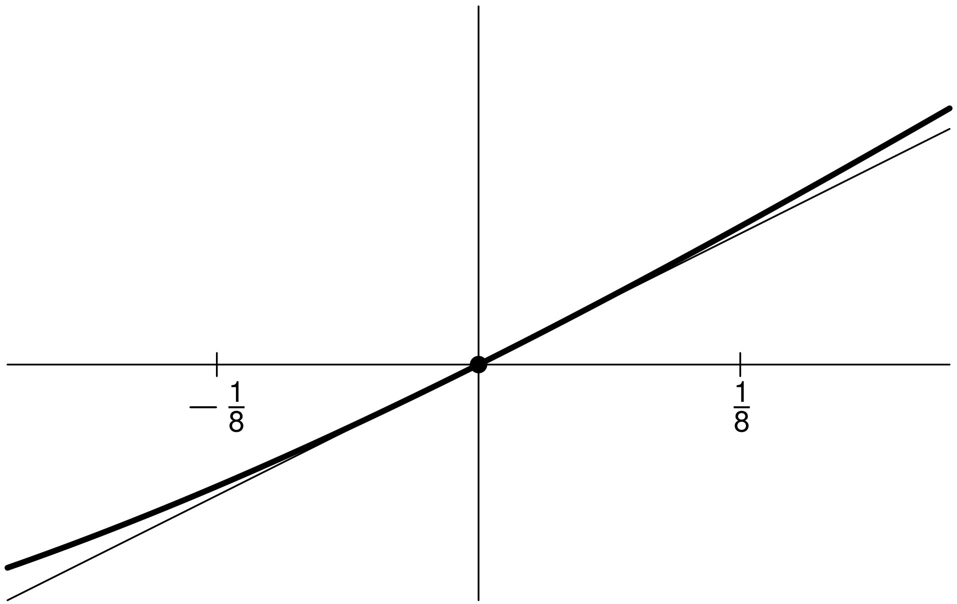

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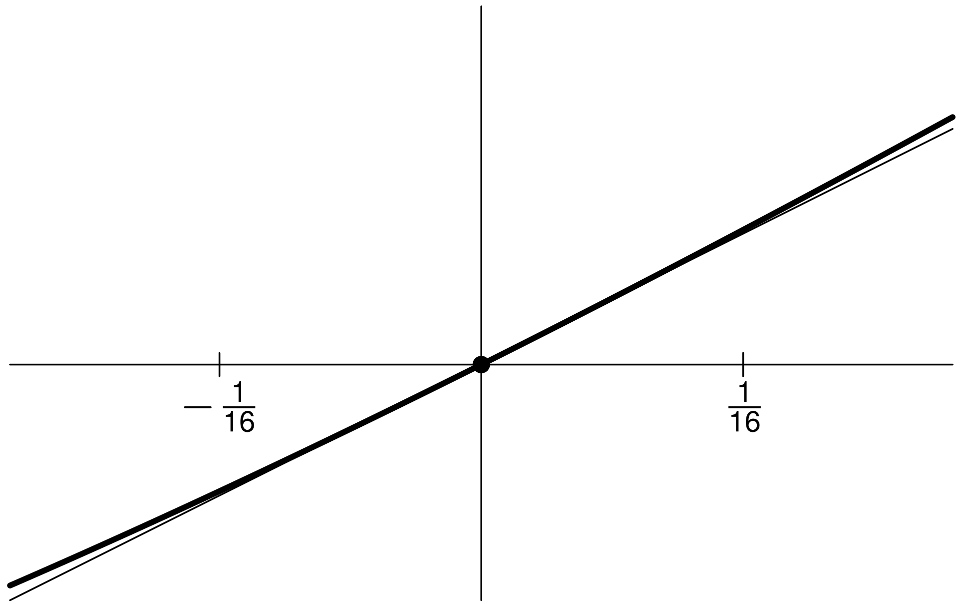

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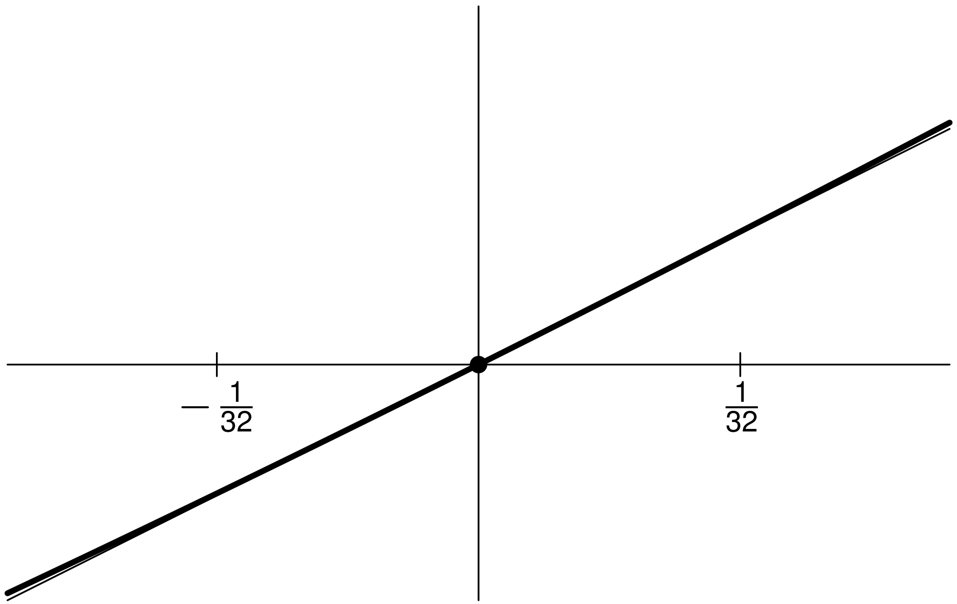

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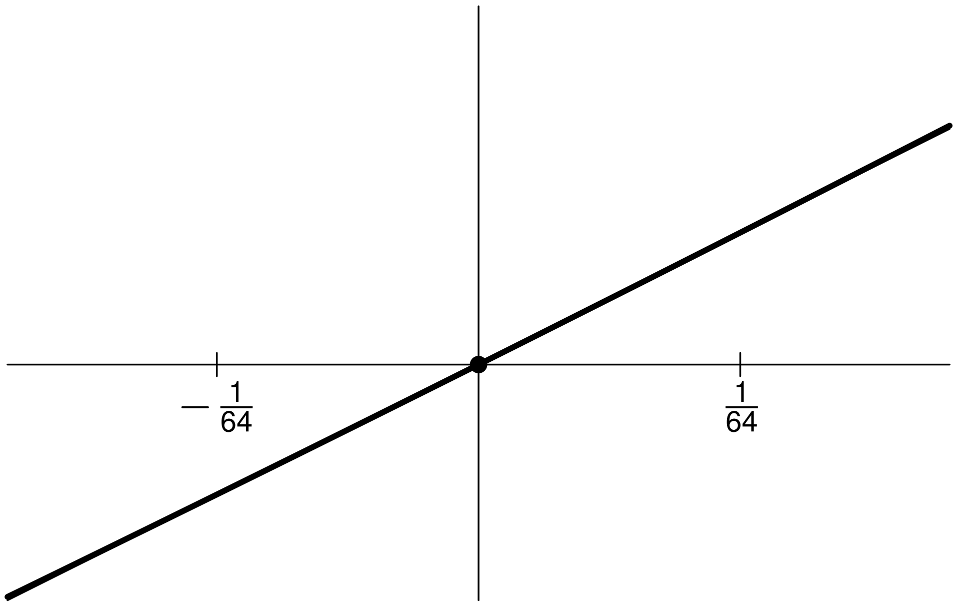

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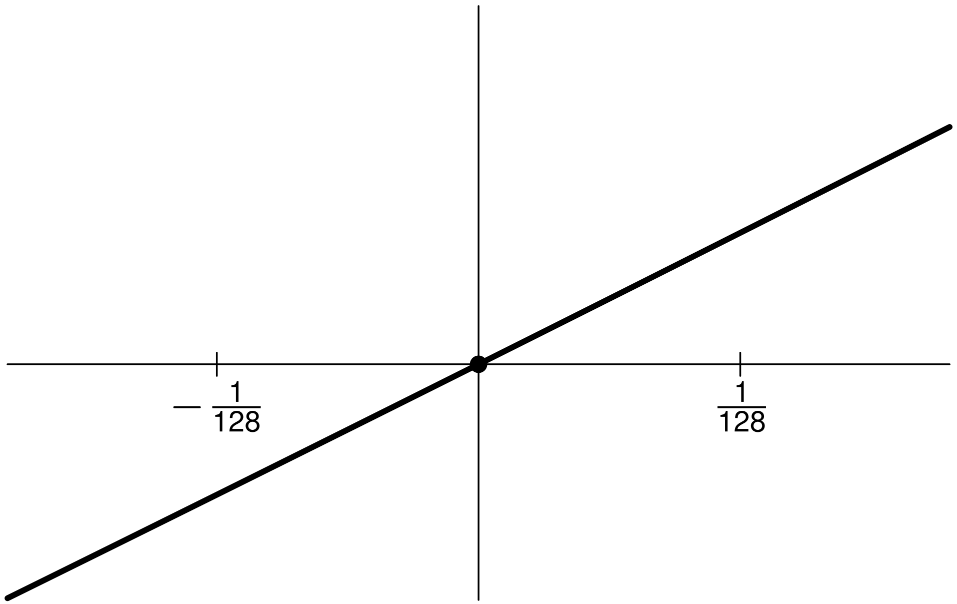

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\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n} \\
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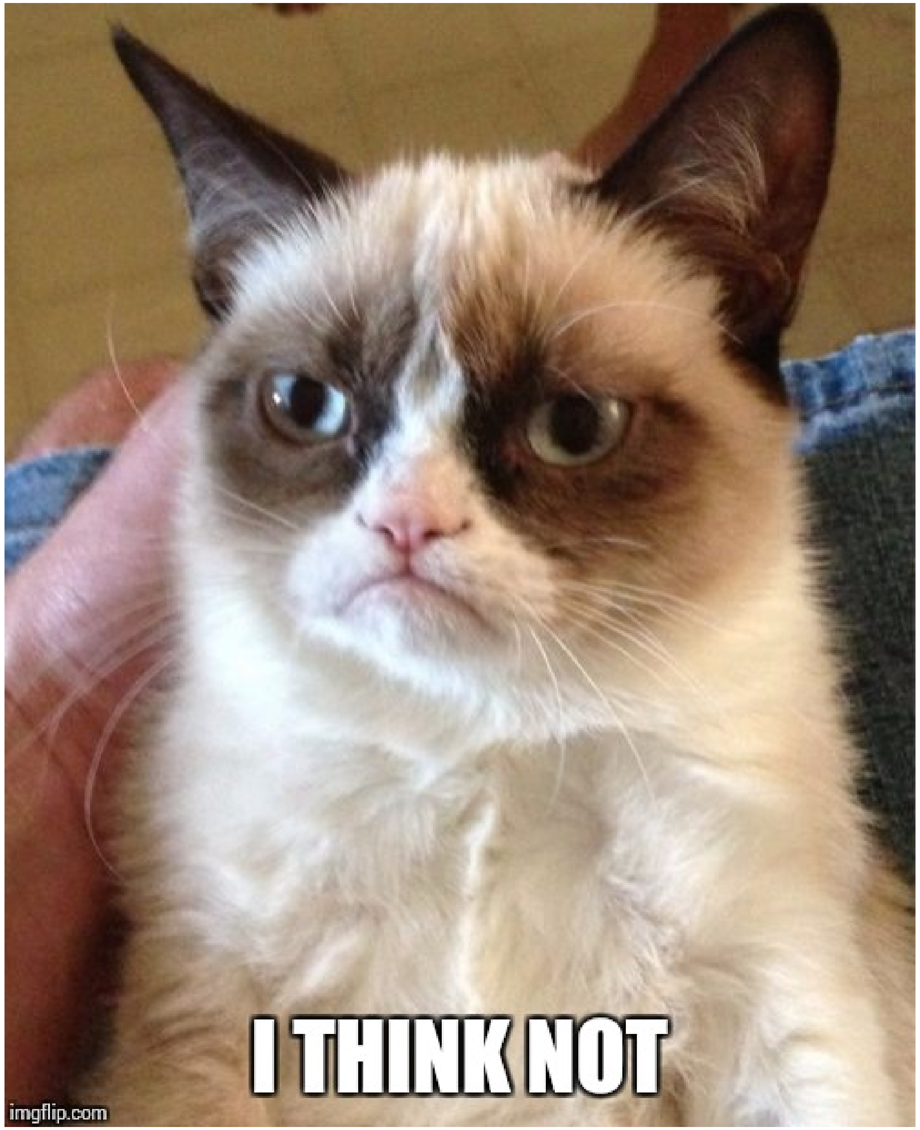
This function (actually a Fourier series) is continuous but is not differentiable **at any point!**


I THINK NOT
imgflip.com

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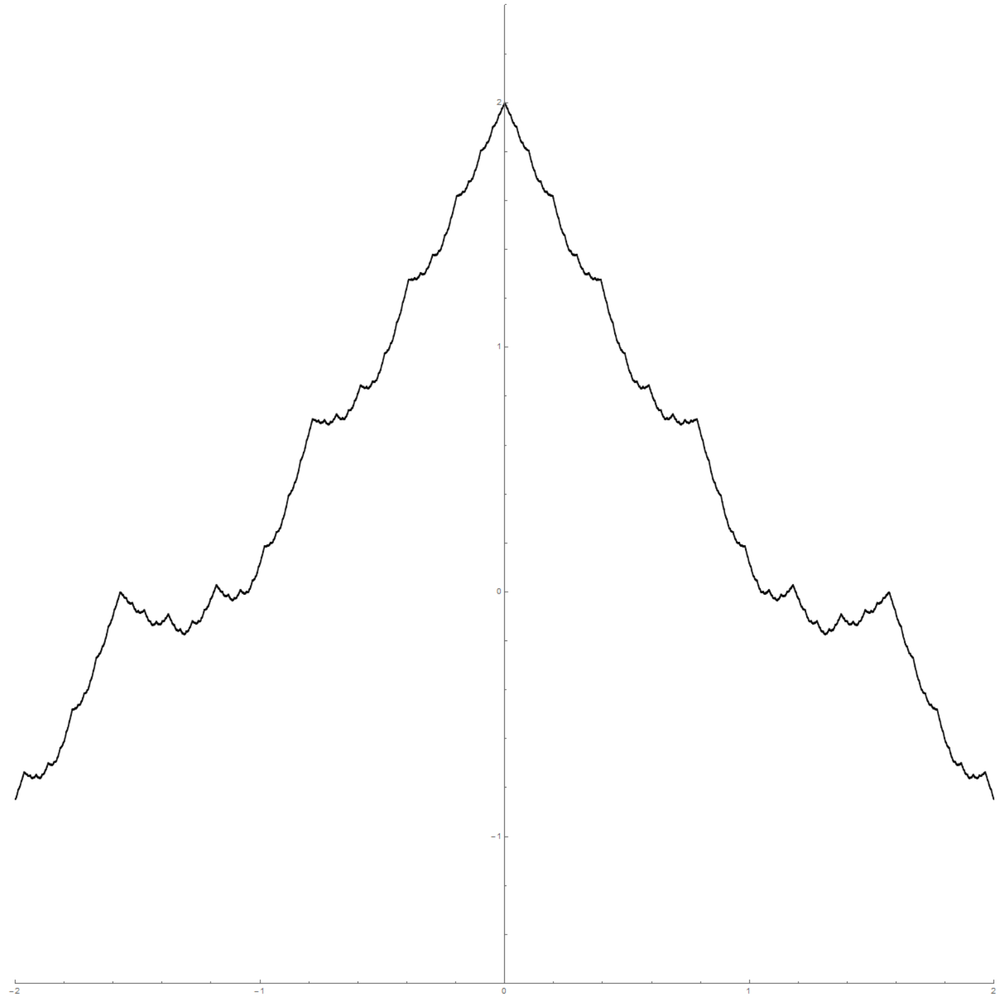

The graph looks a bit like a jagged mountain – it is ‘infinitely’ rough.

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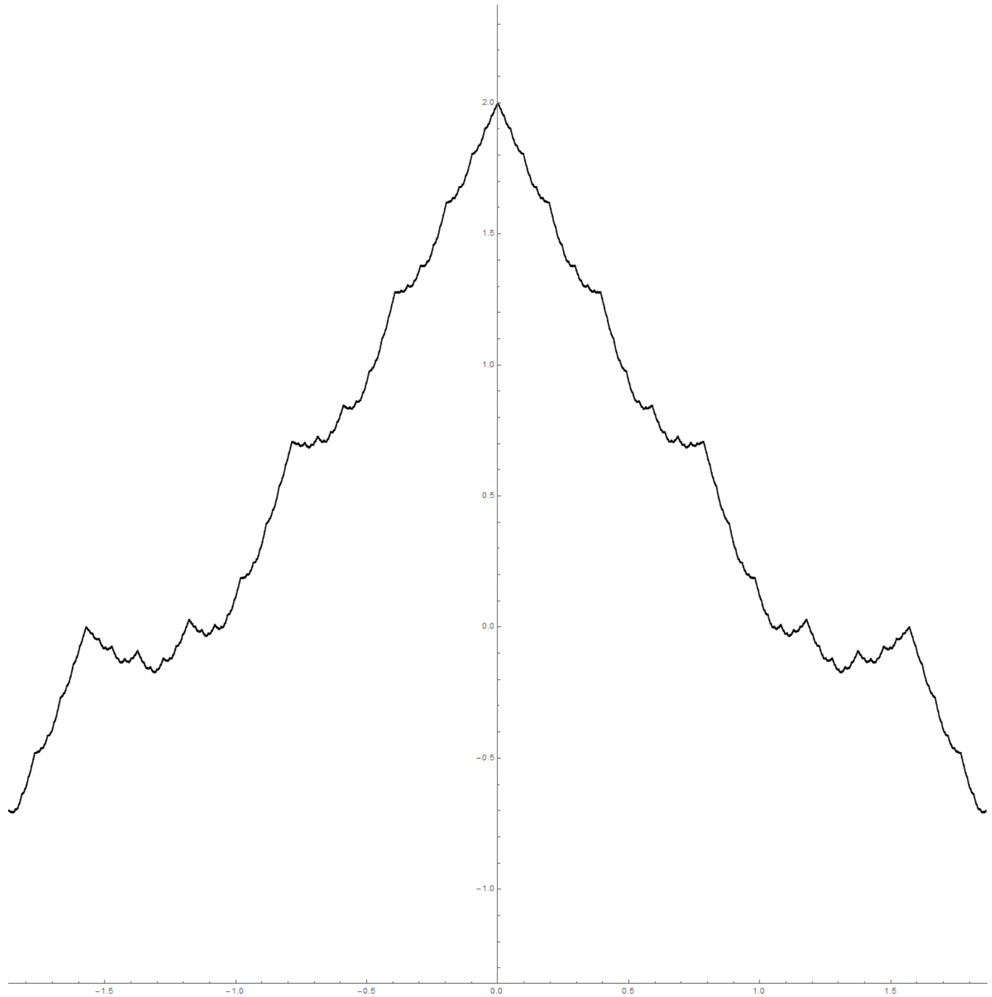

We zoom in to the point $(0, 2)$ on the graph (when $x = 0$, $y = 2$).

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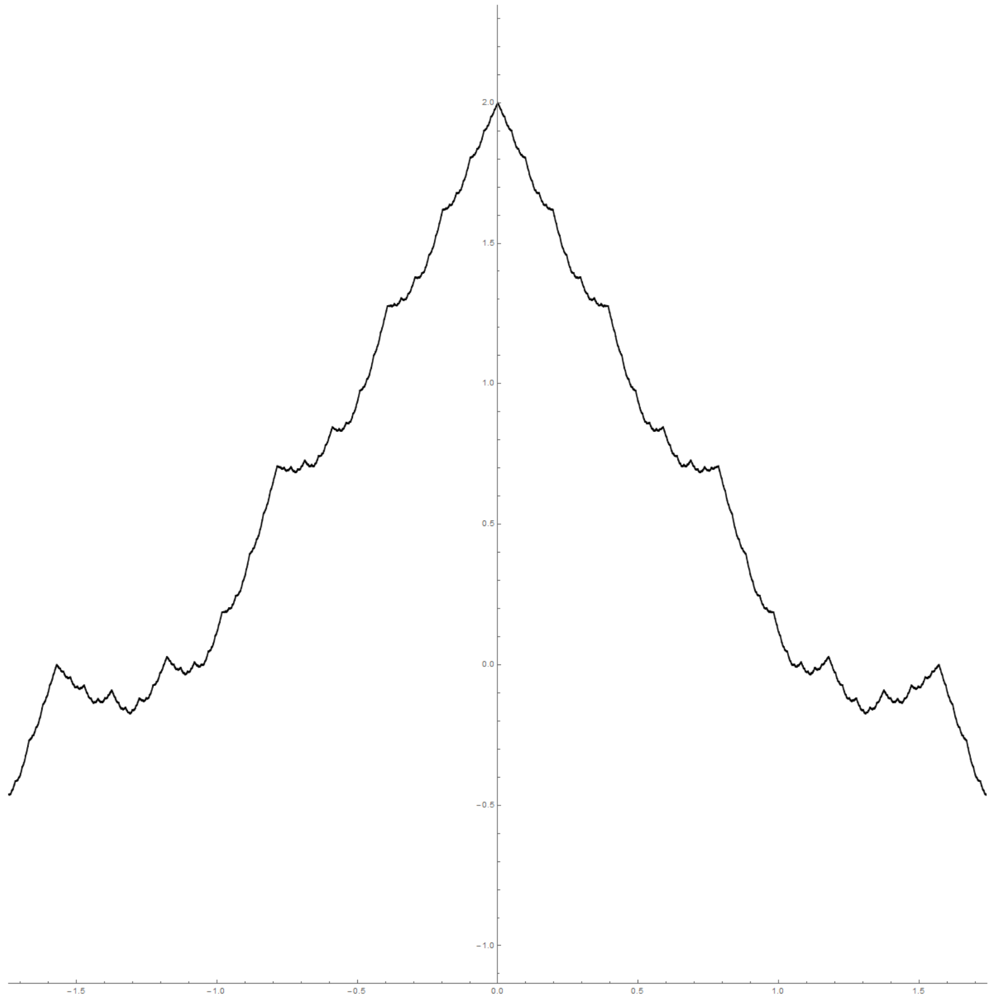

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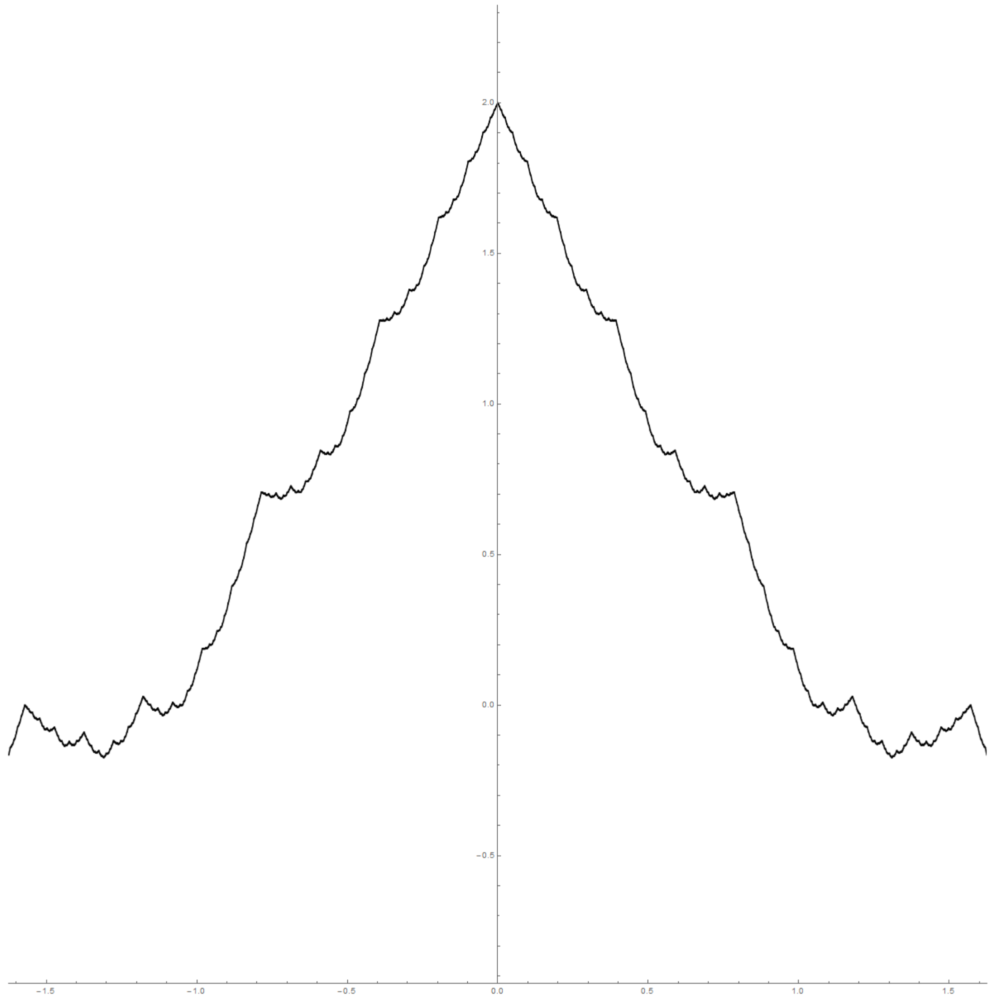

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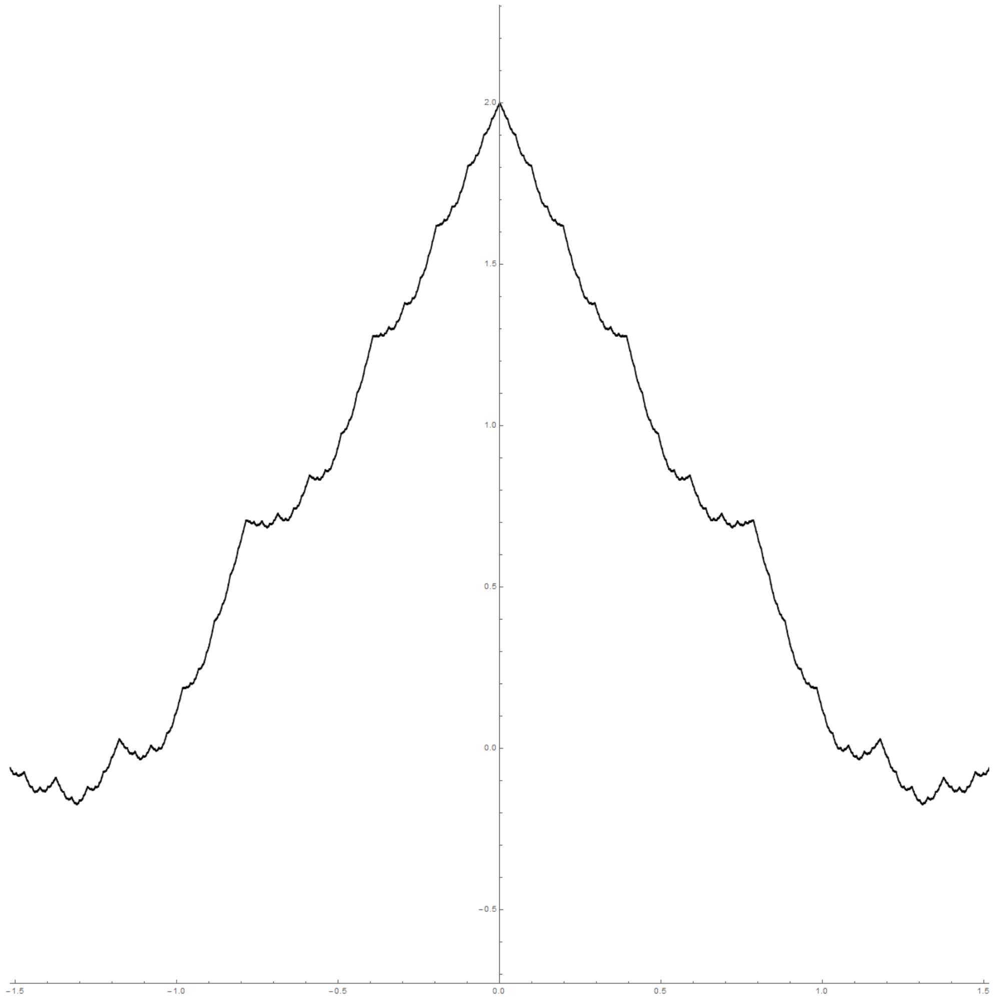

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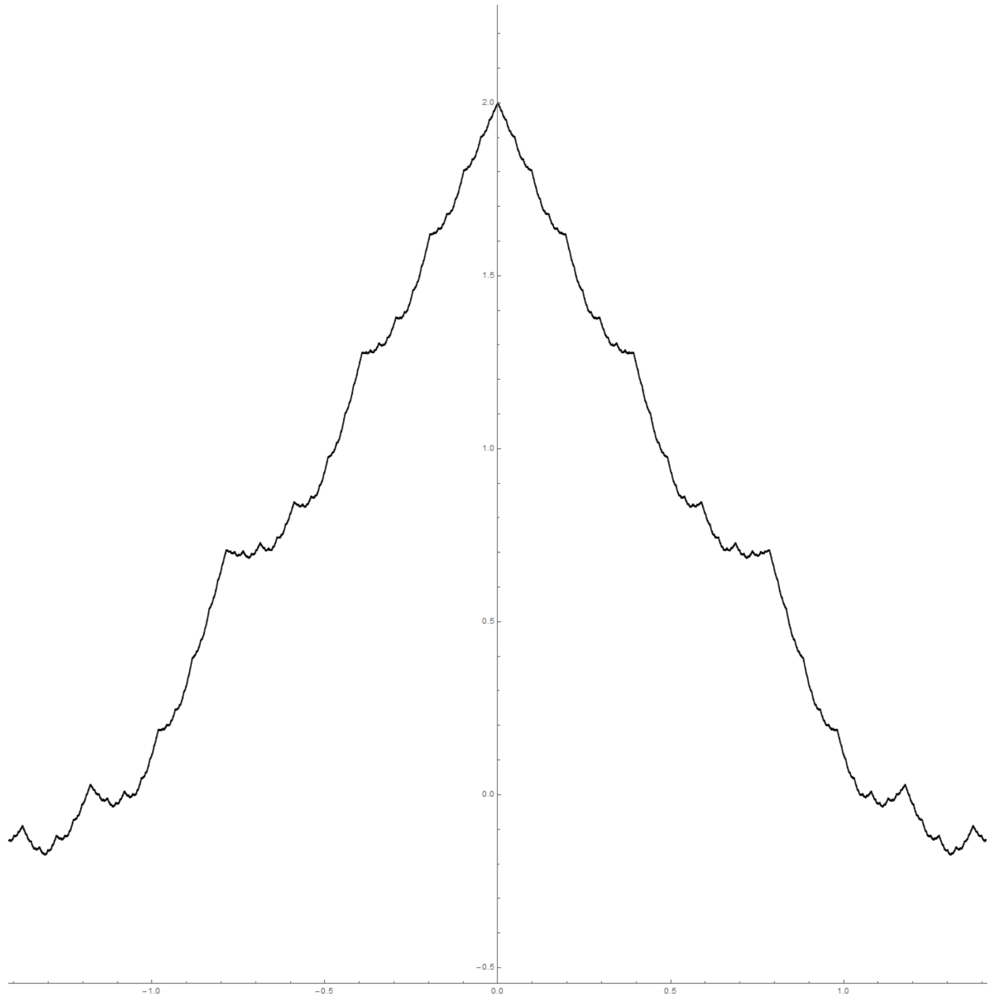

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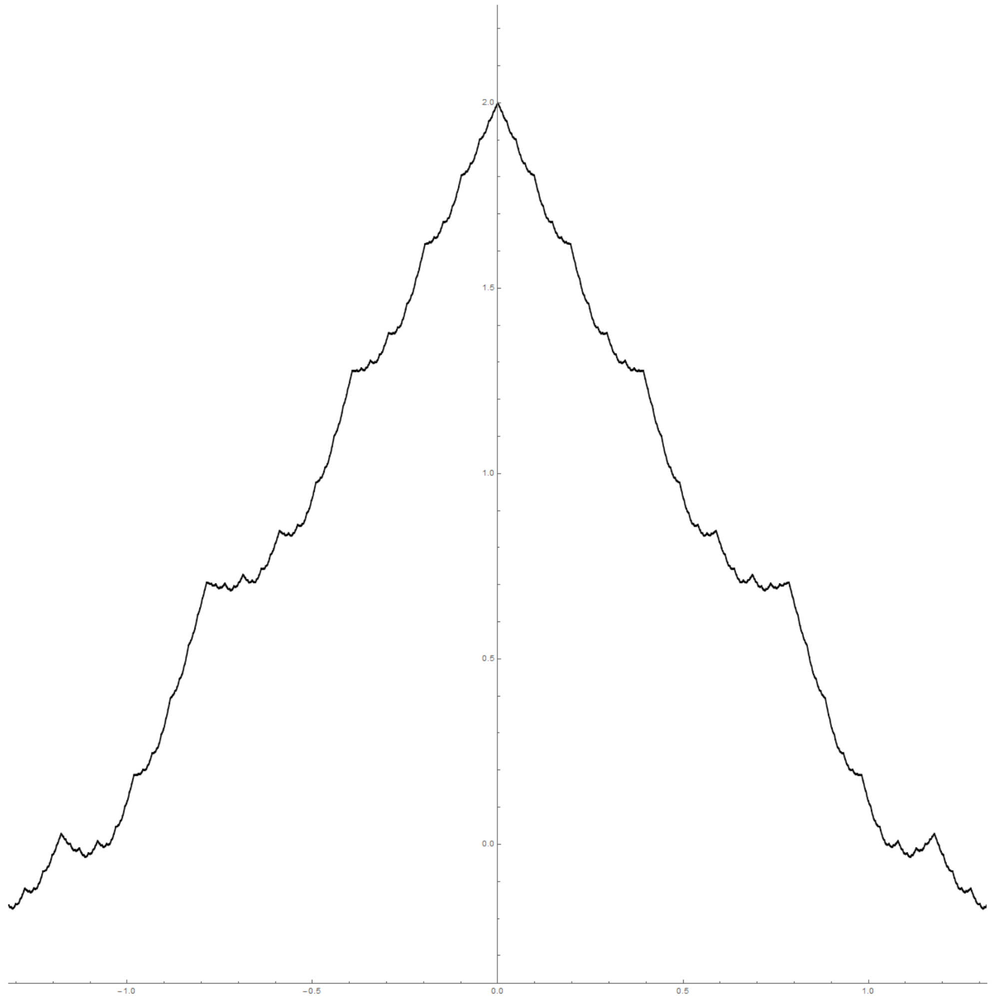

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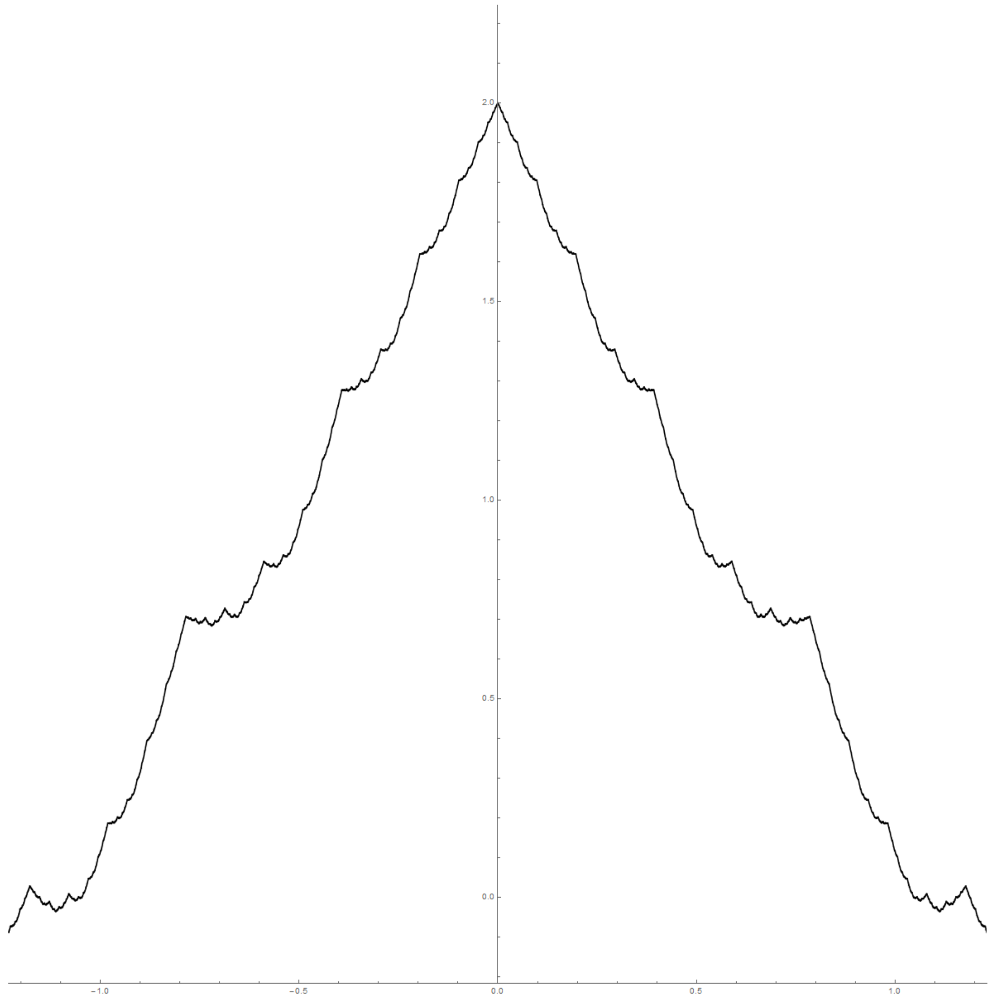

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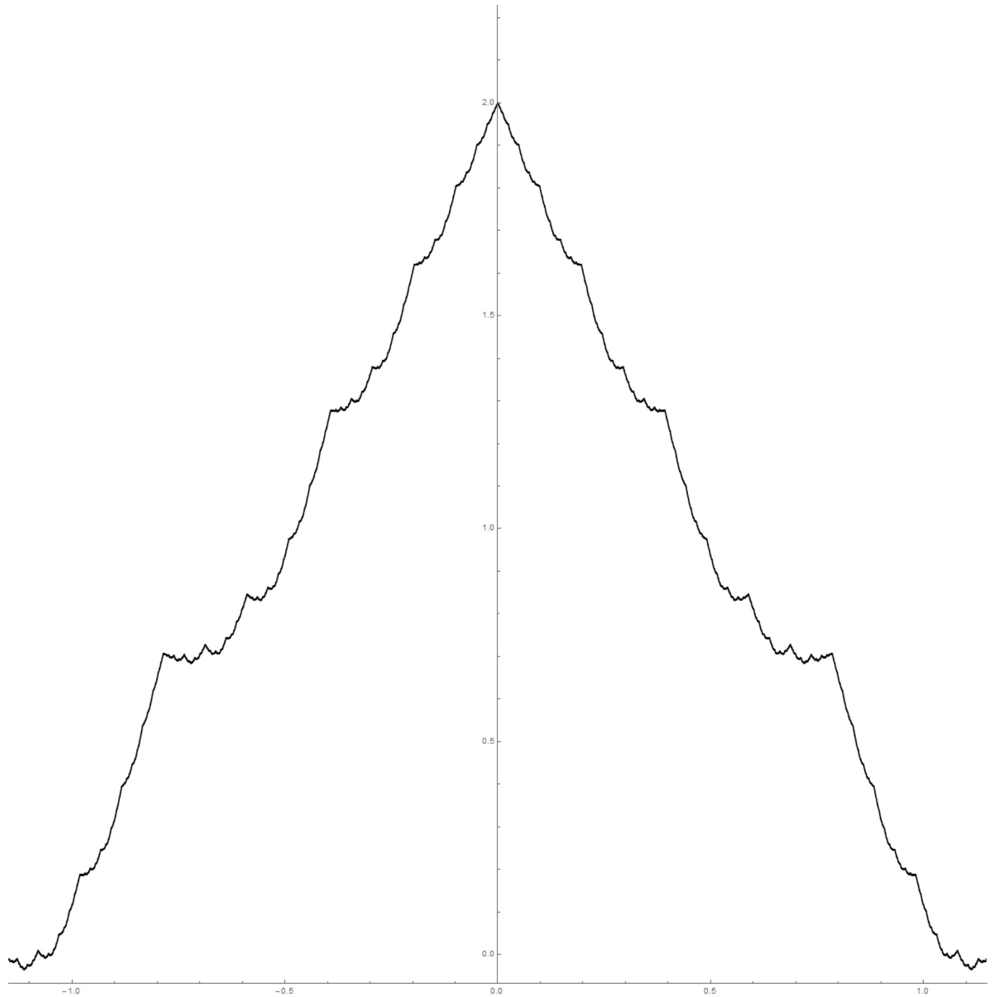

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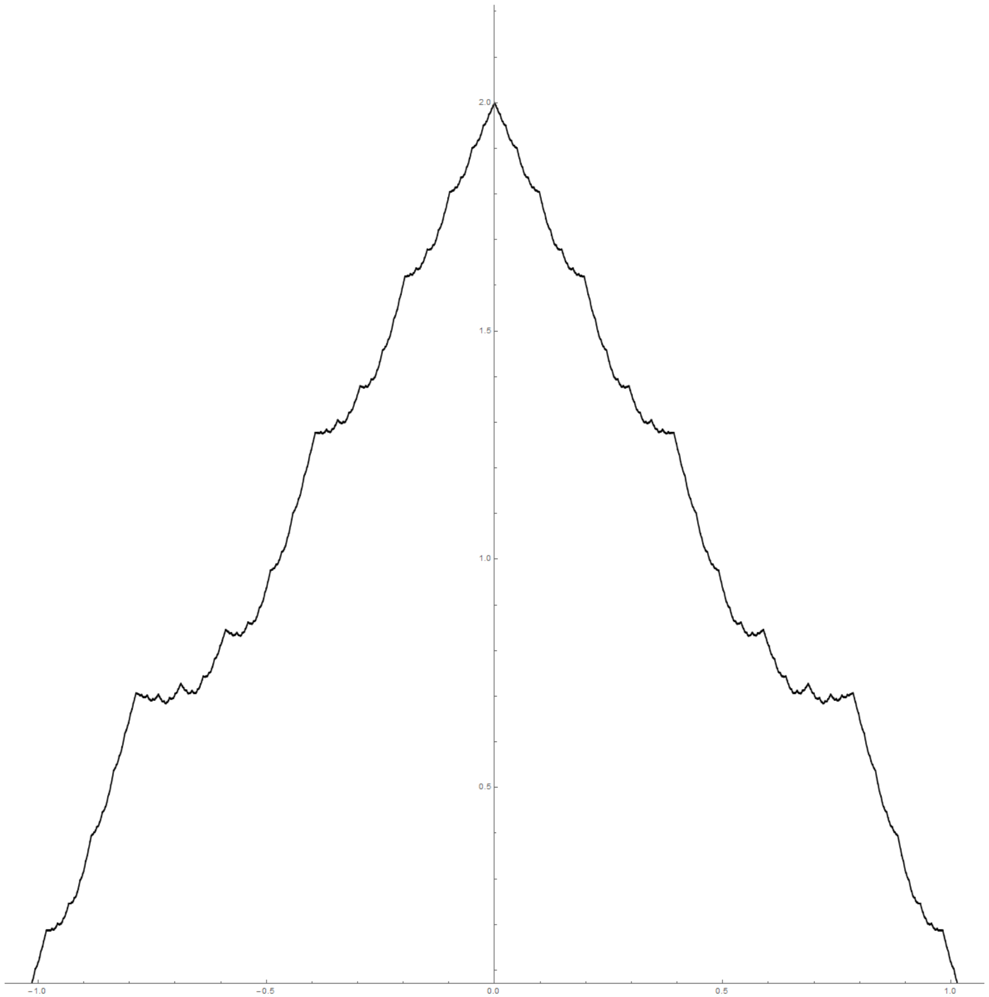

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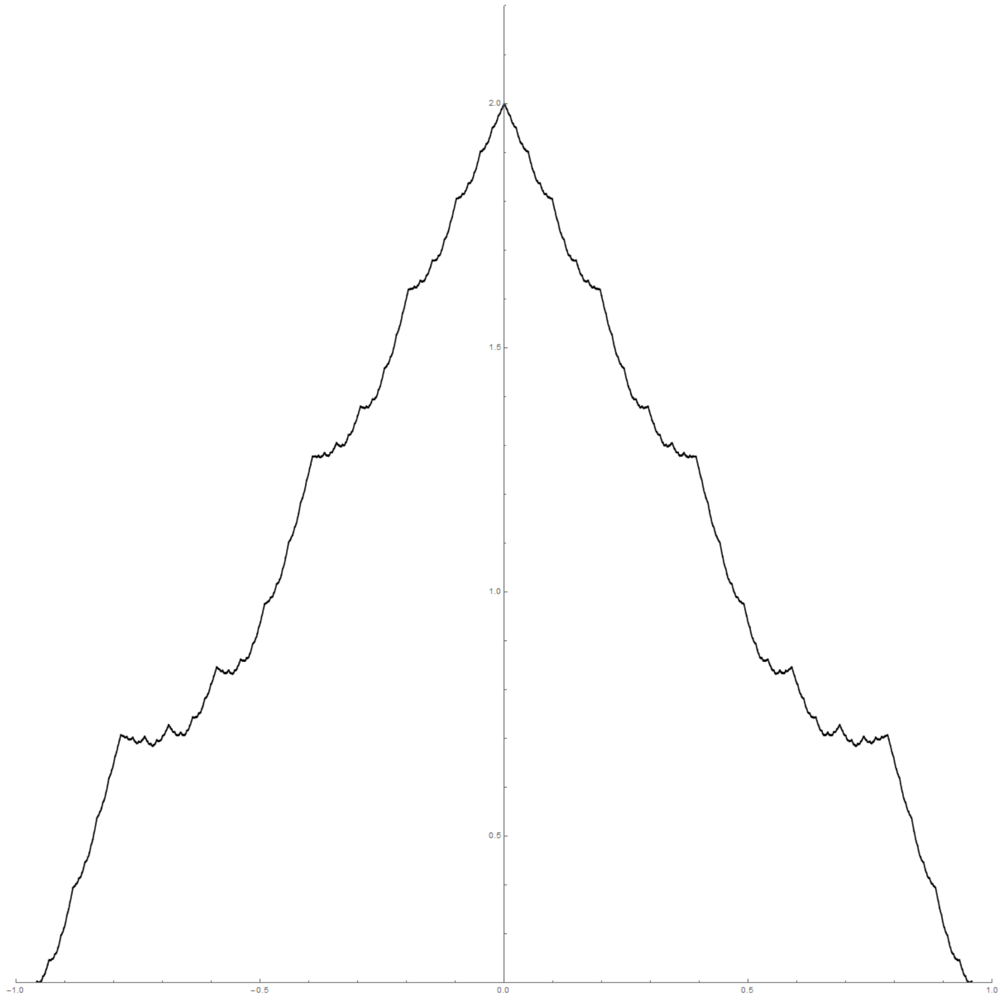

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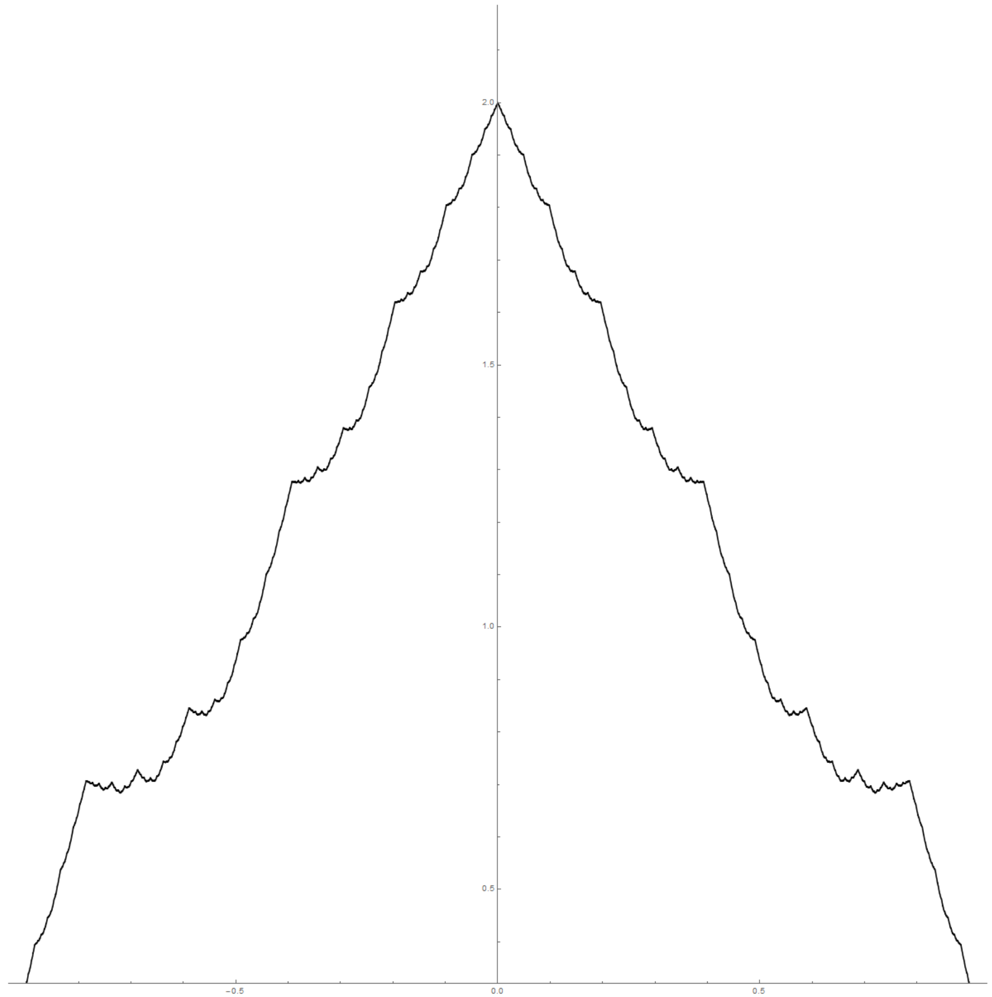

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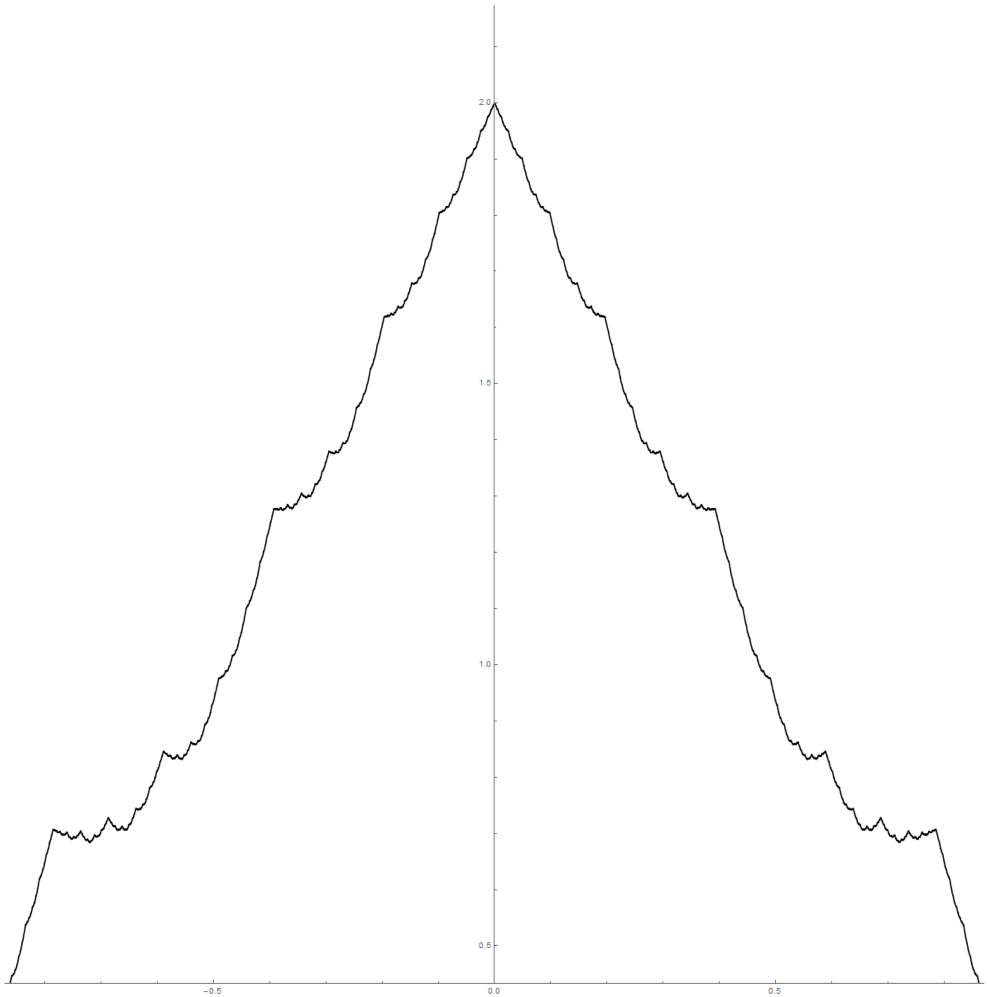

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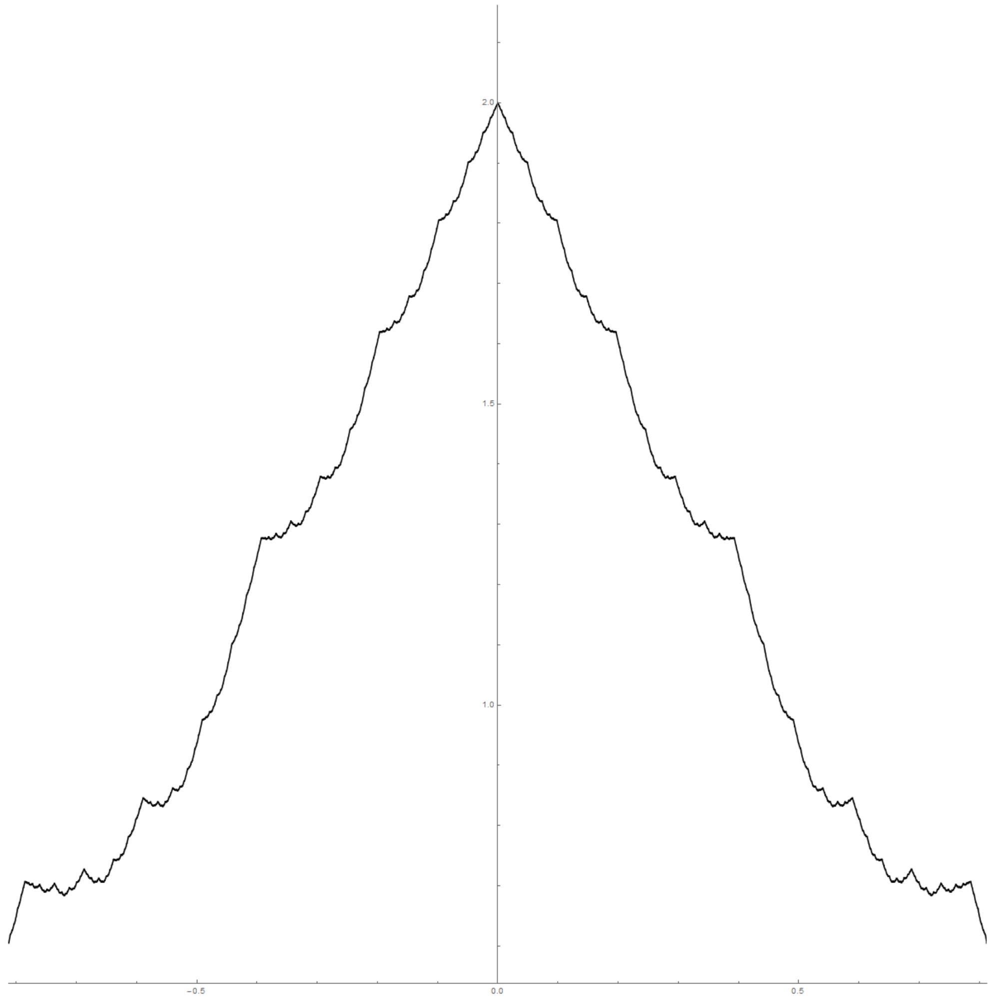

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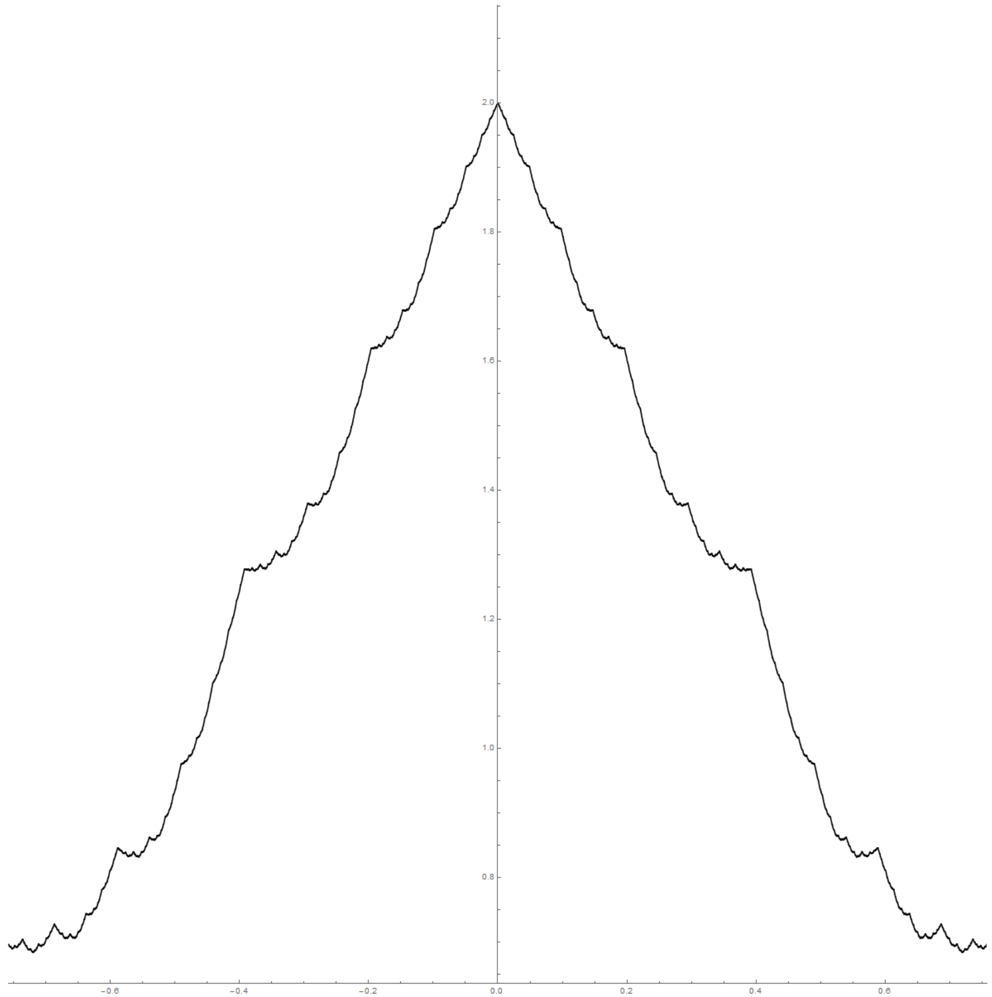

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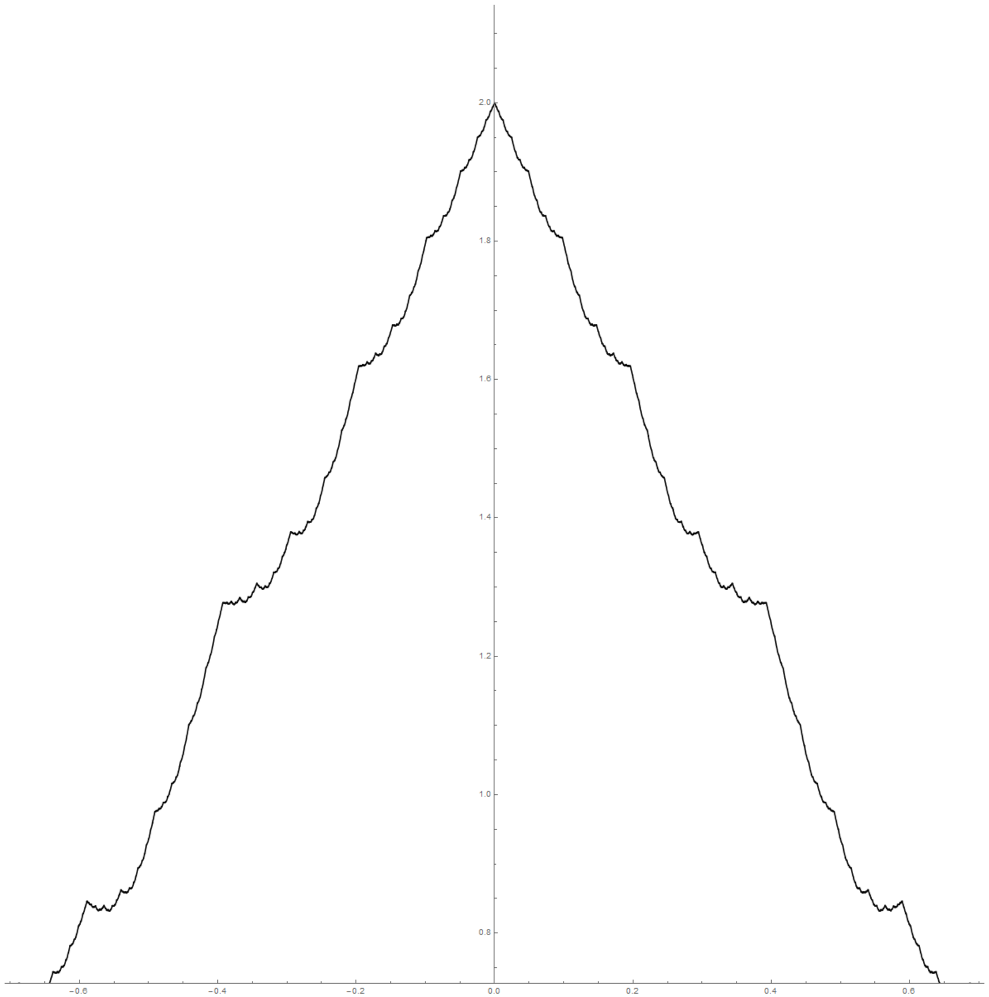

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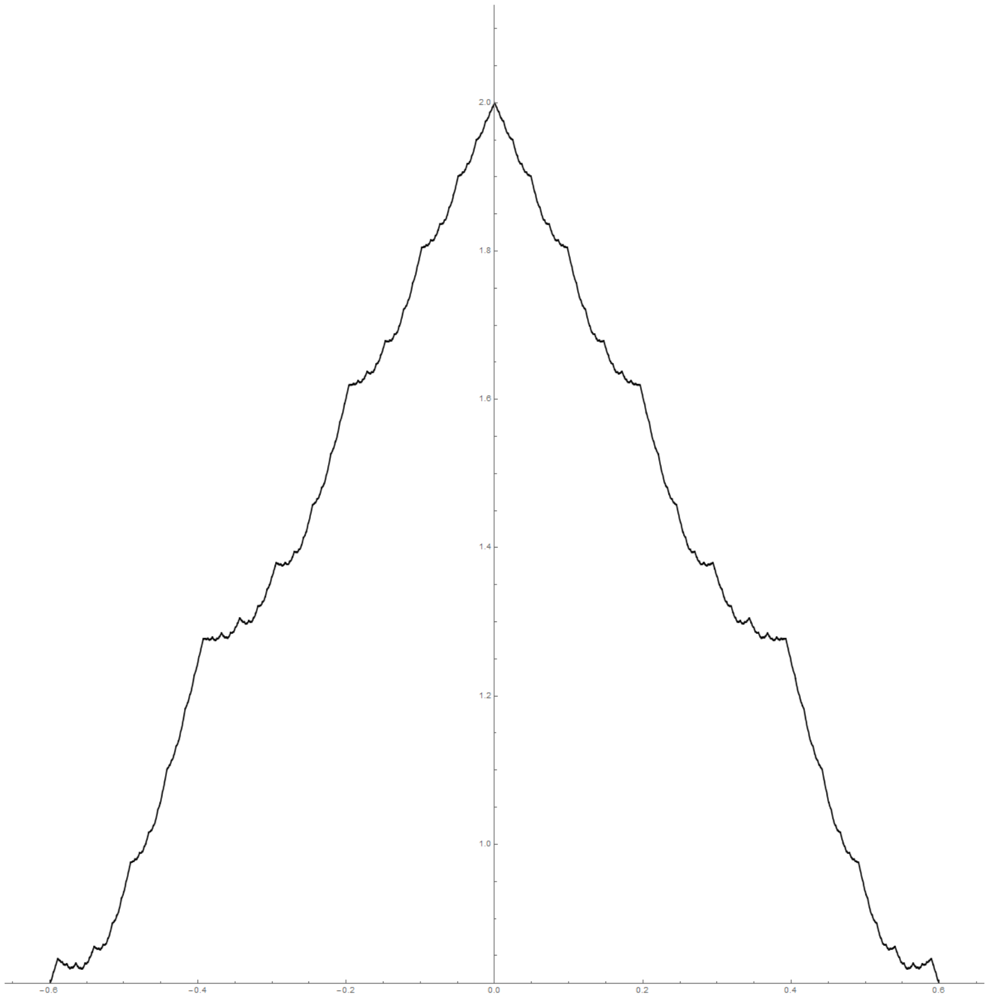

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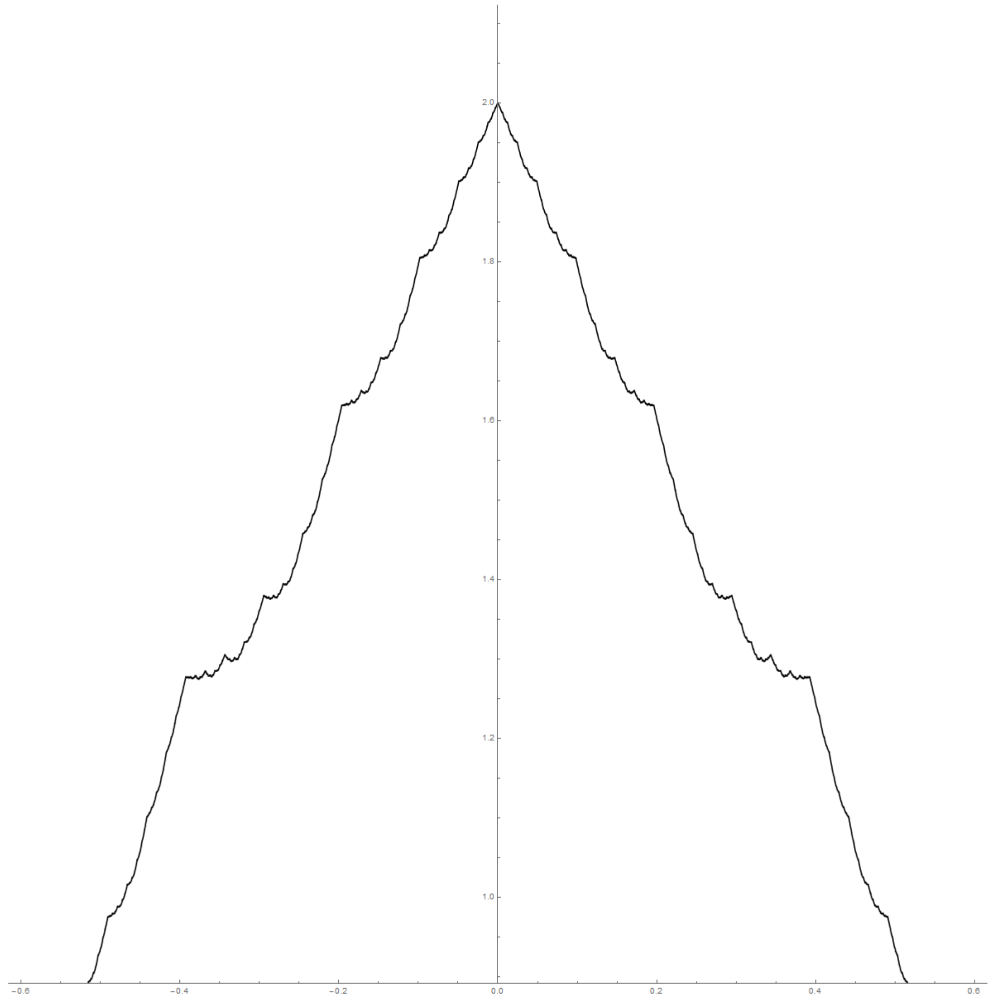

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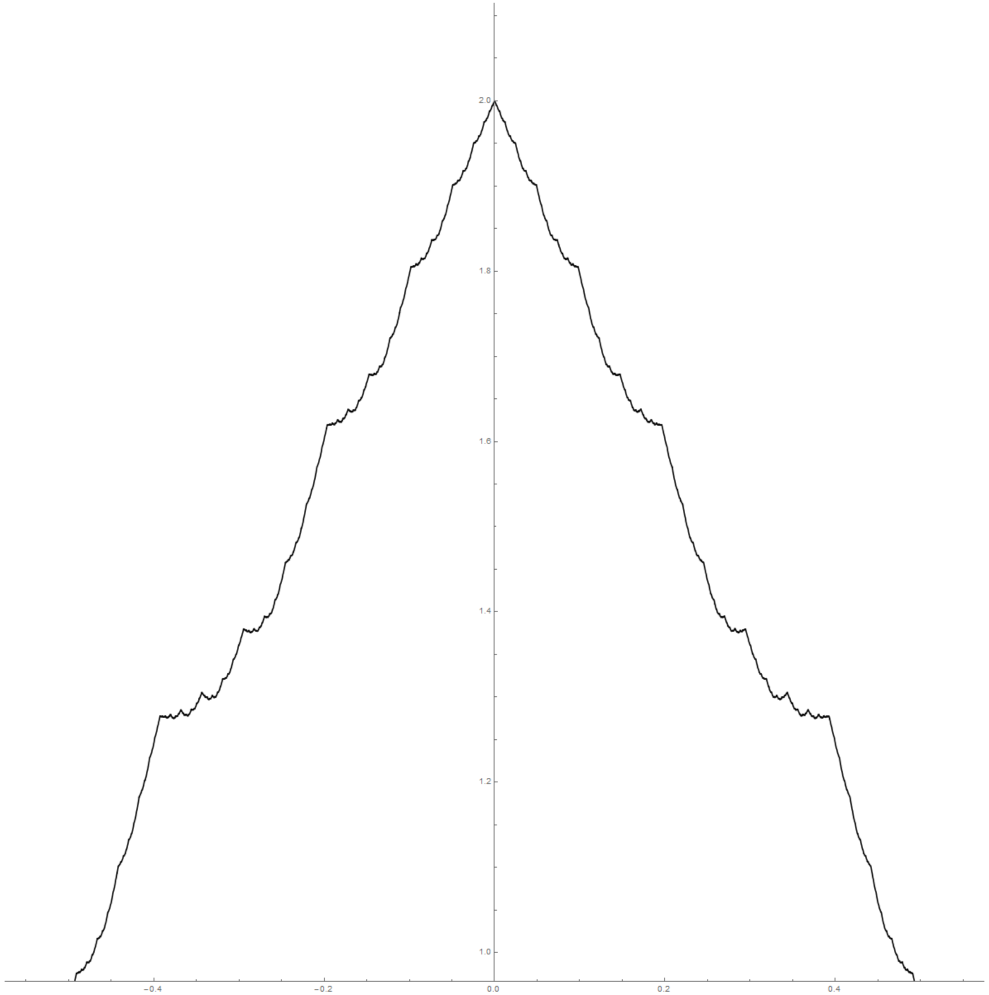

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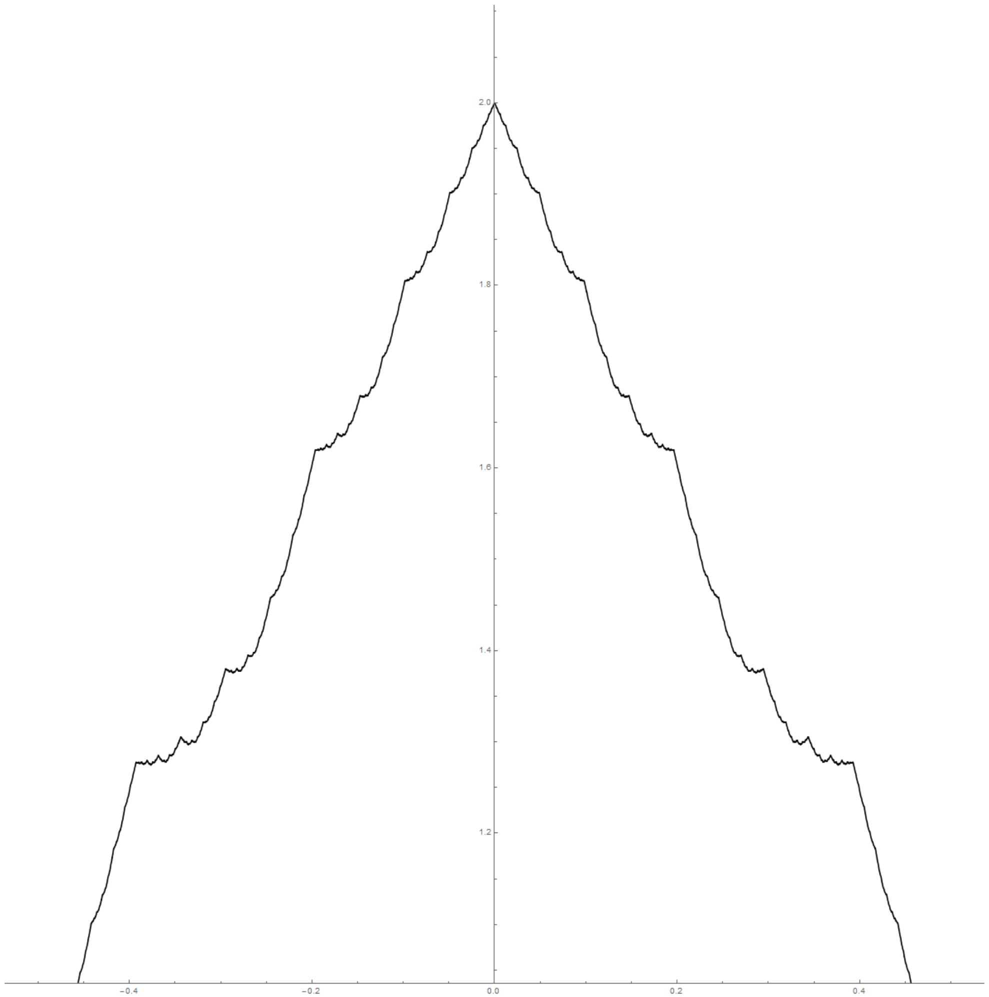

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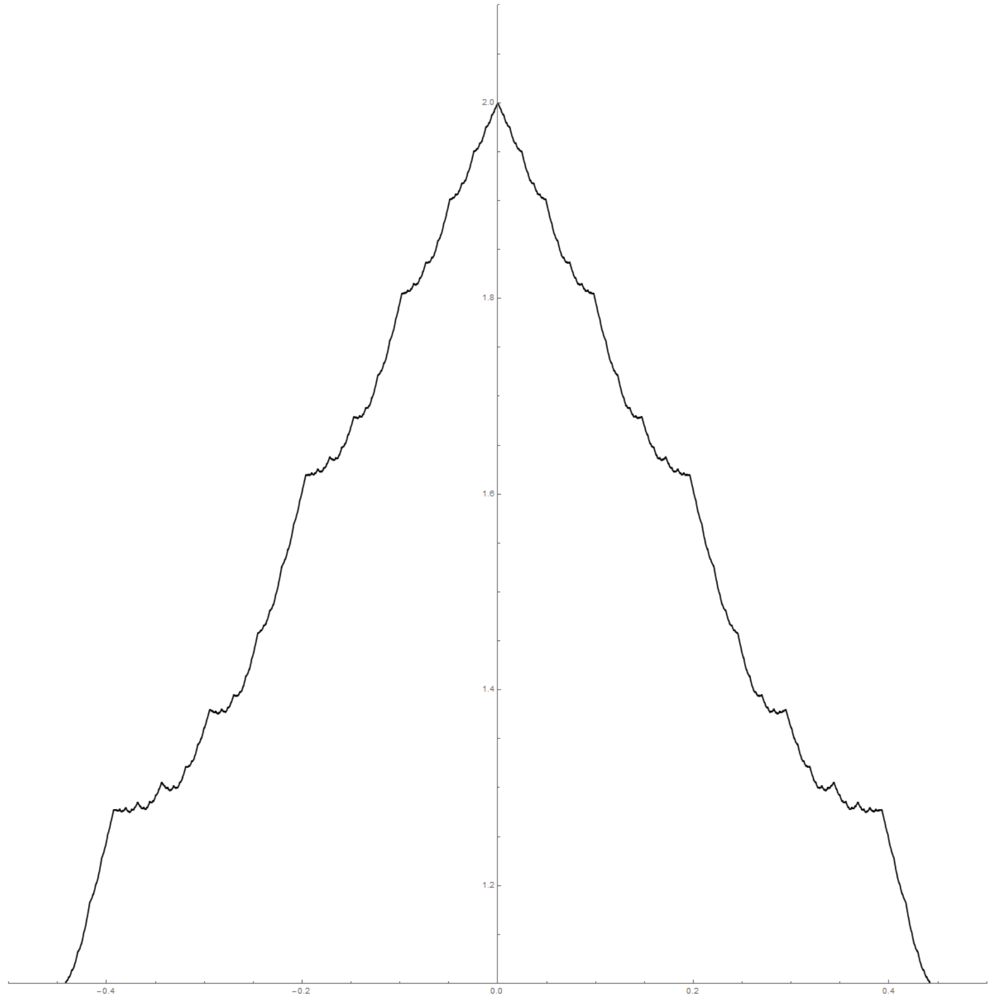

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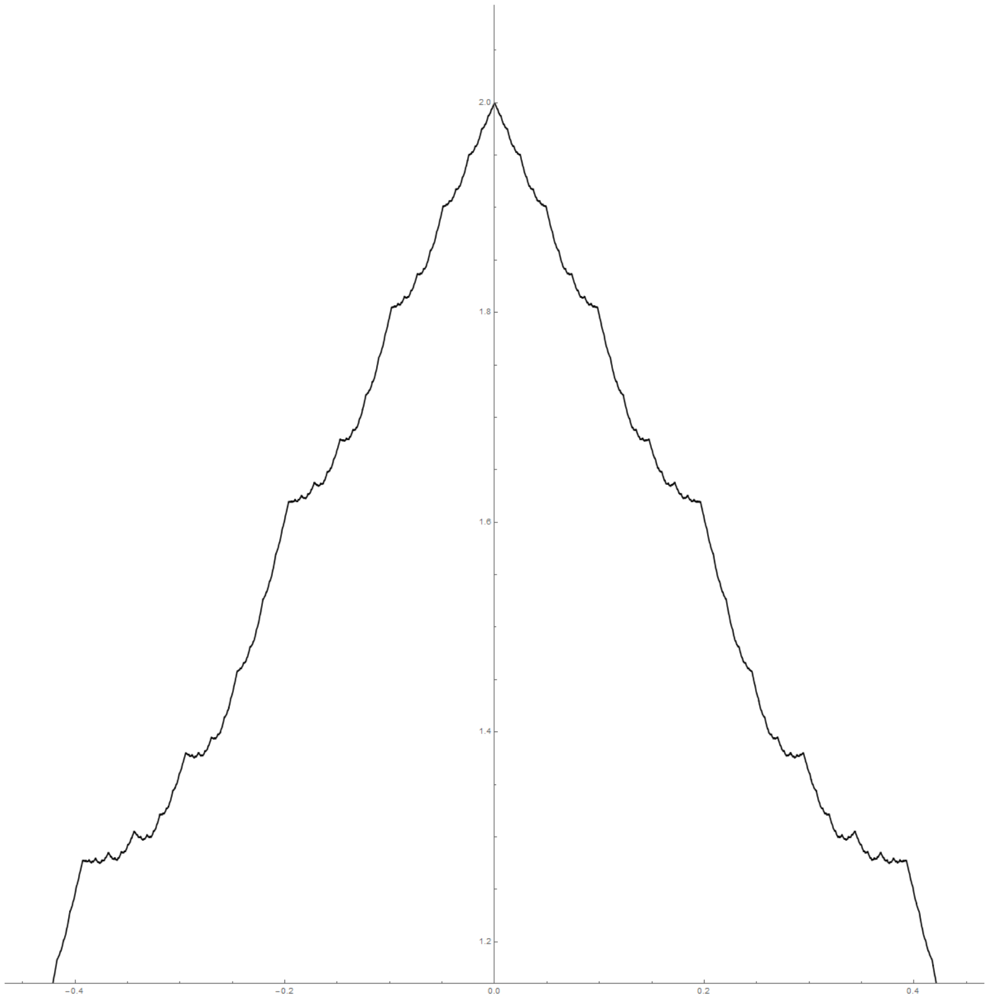

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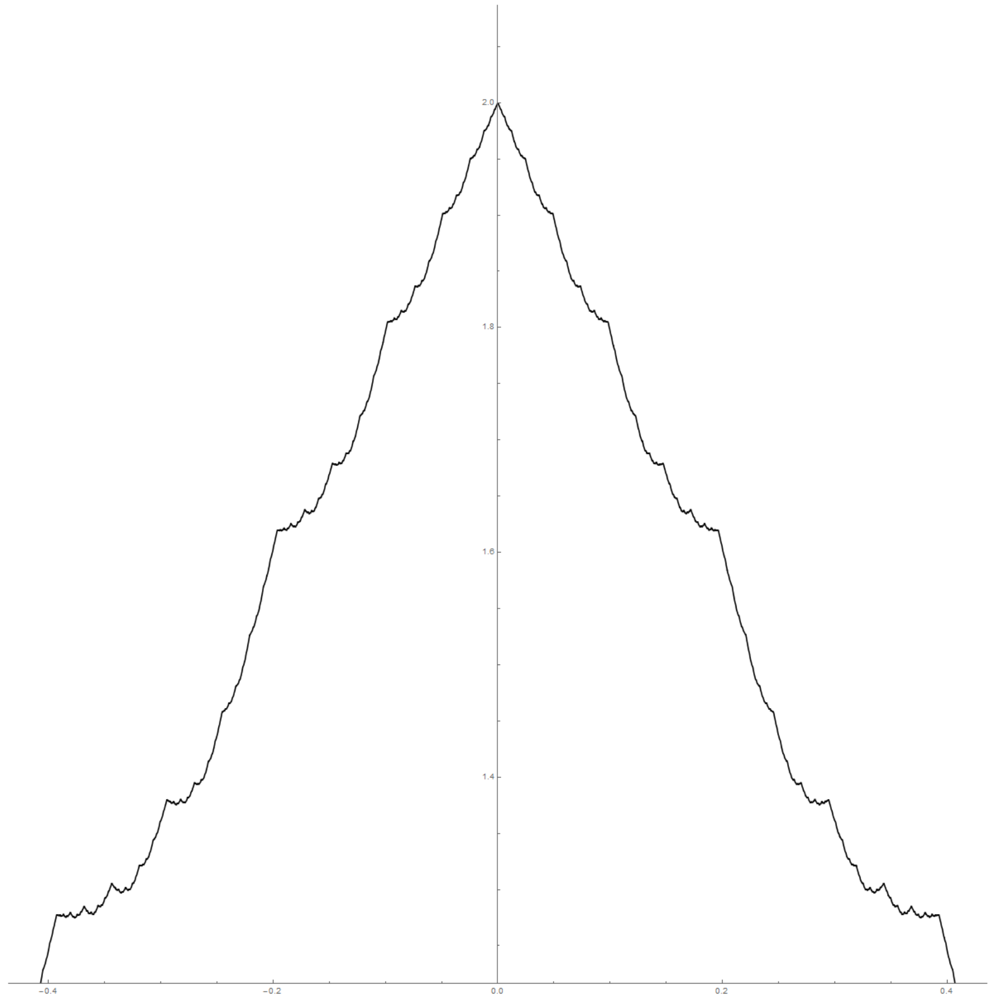

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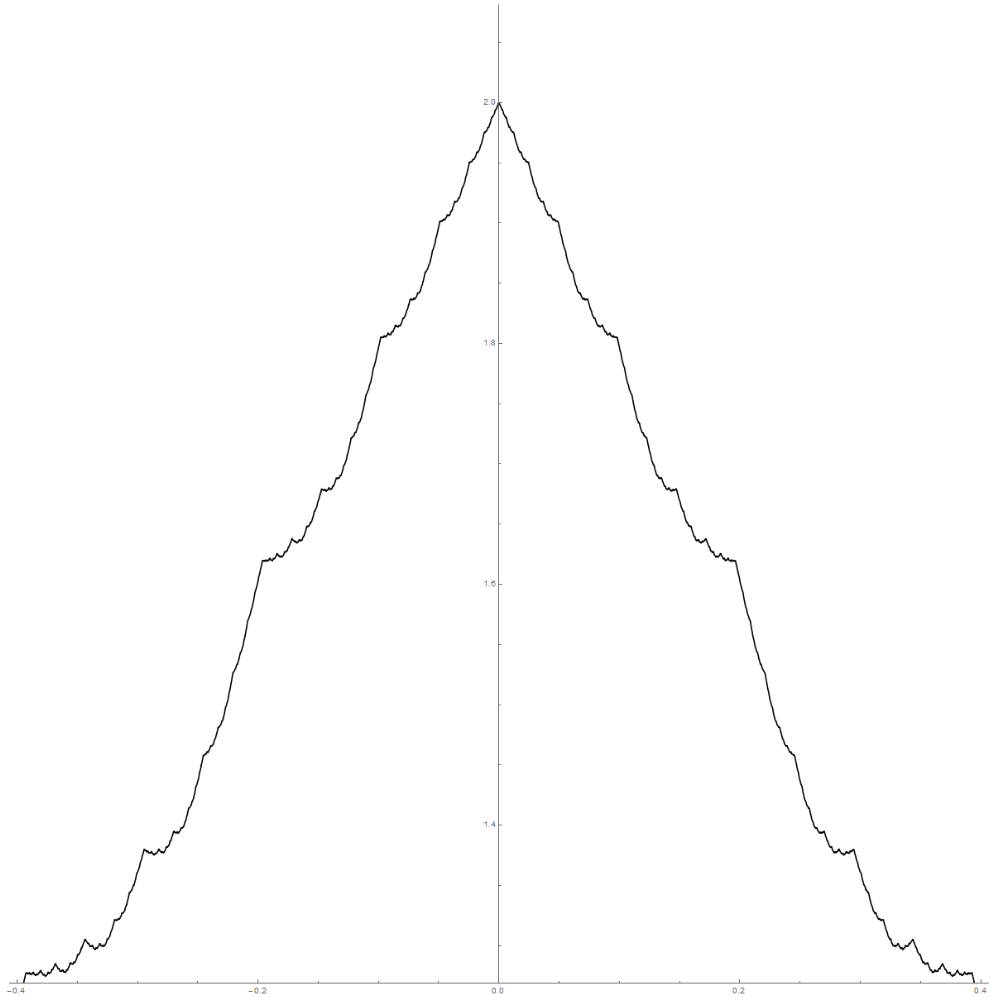

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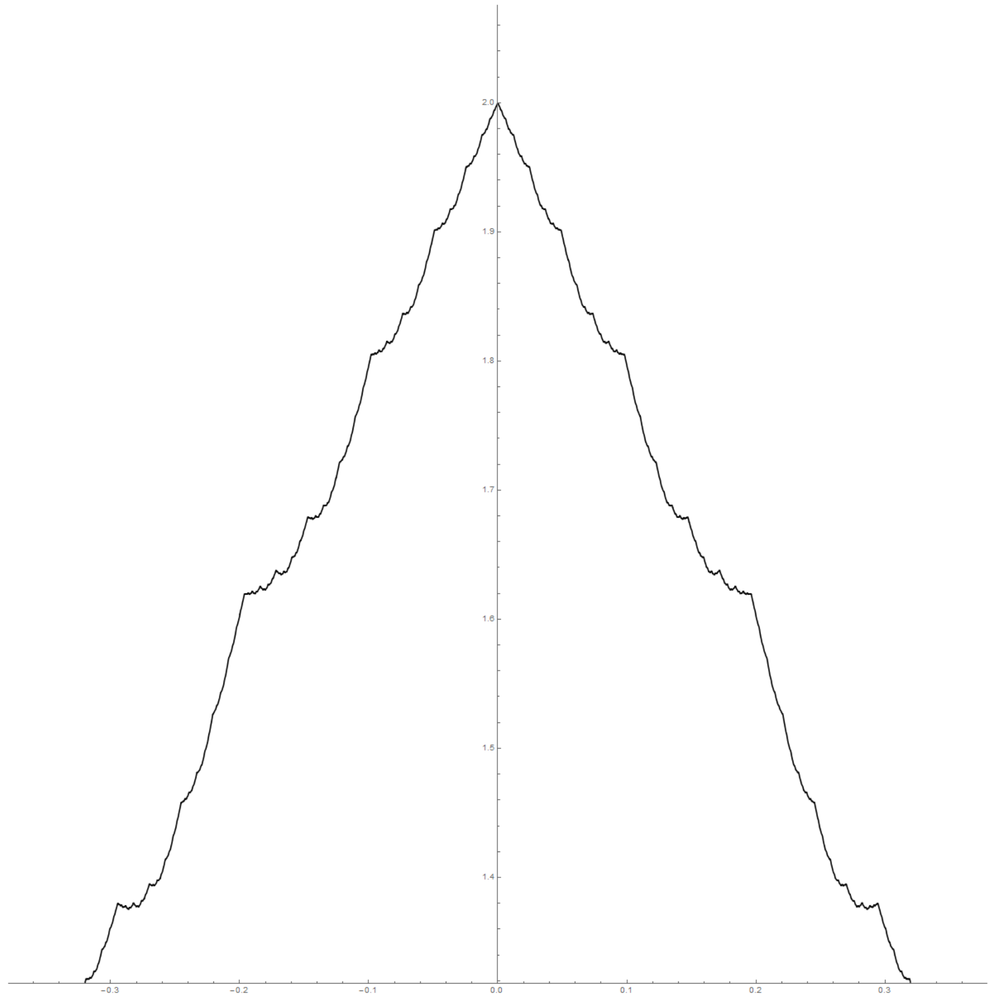

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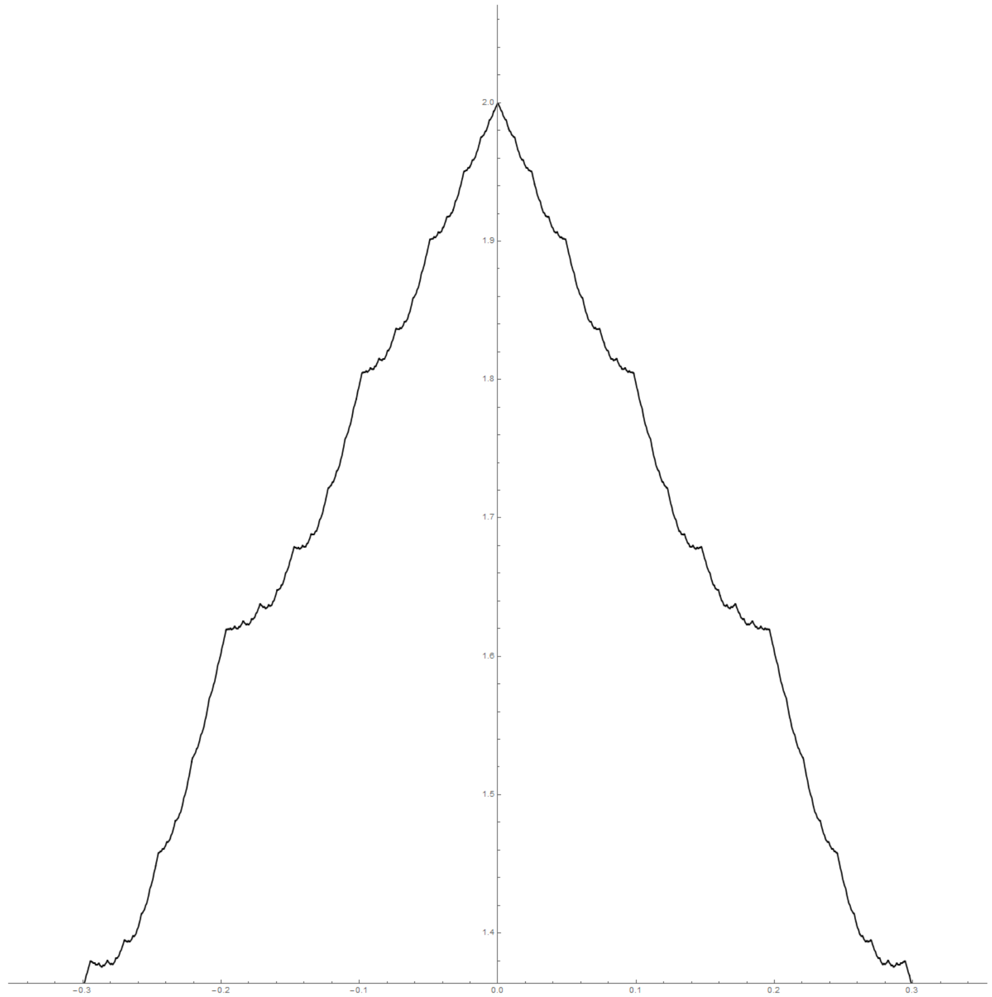

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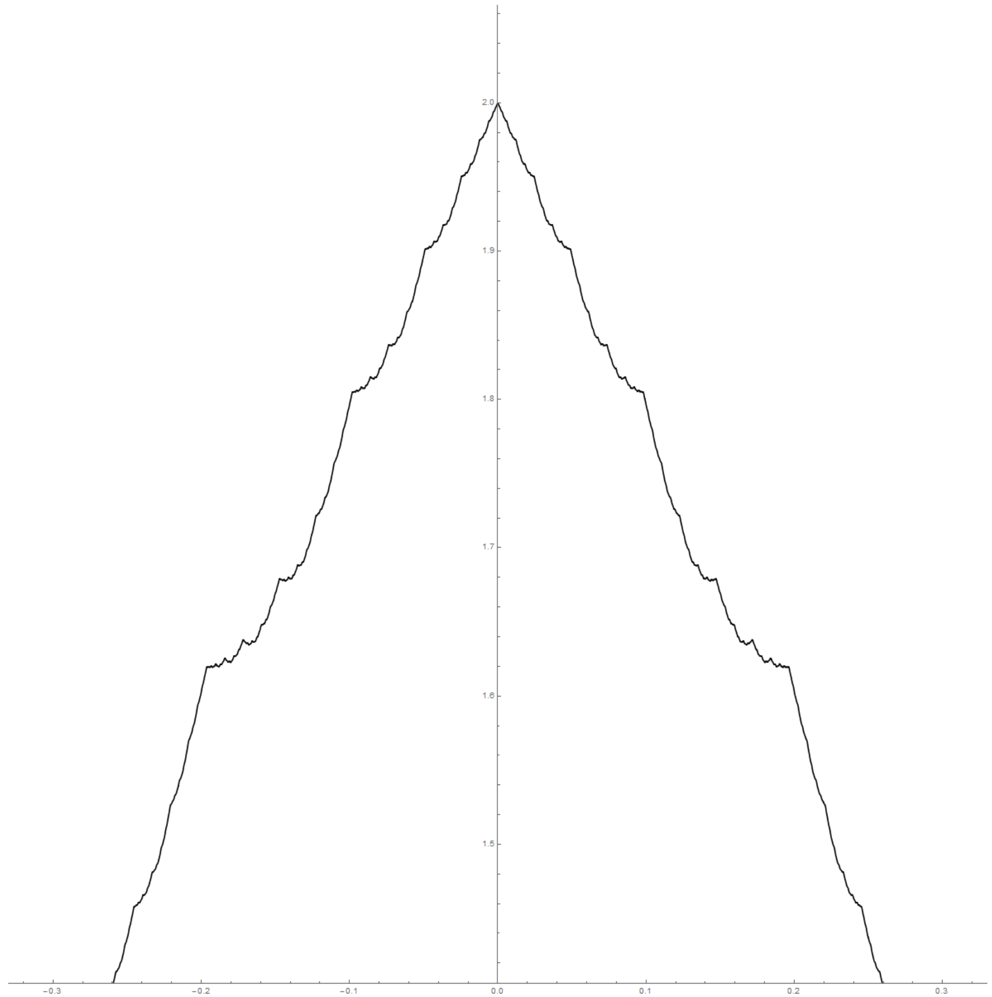

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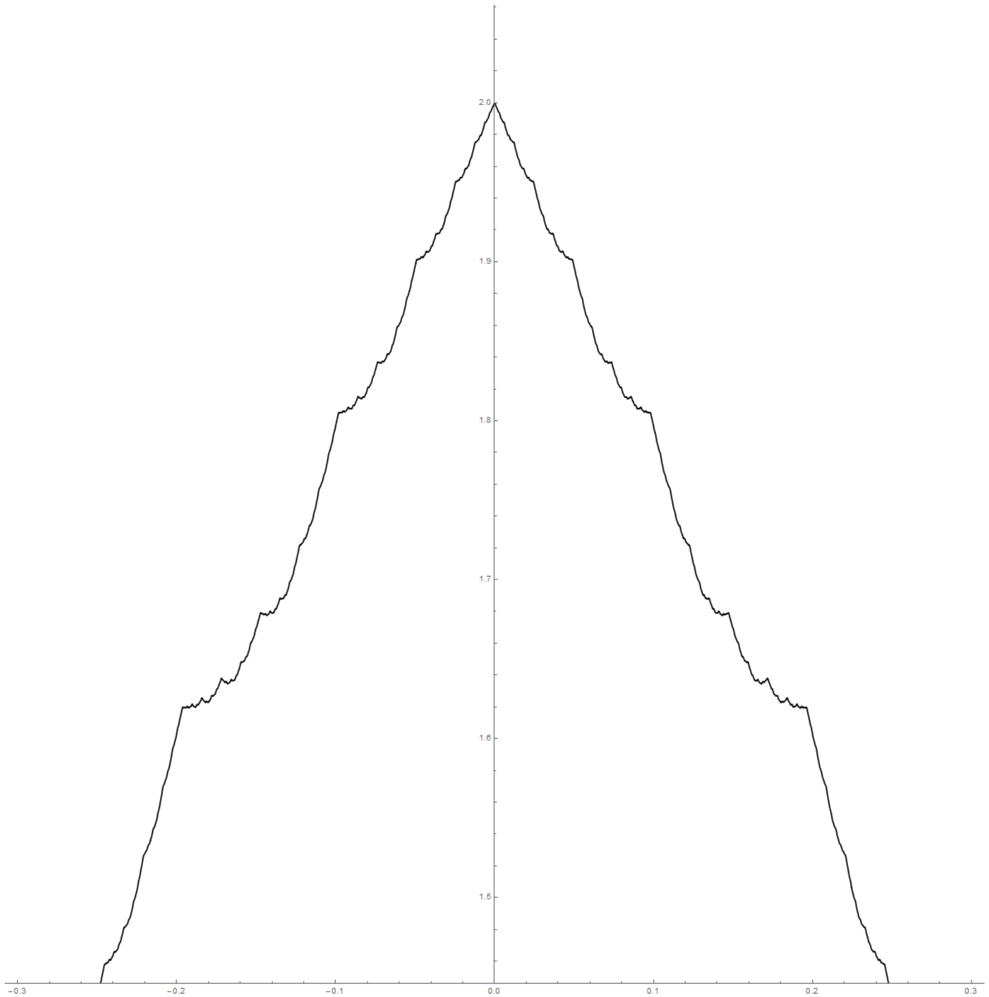

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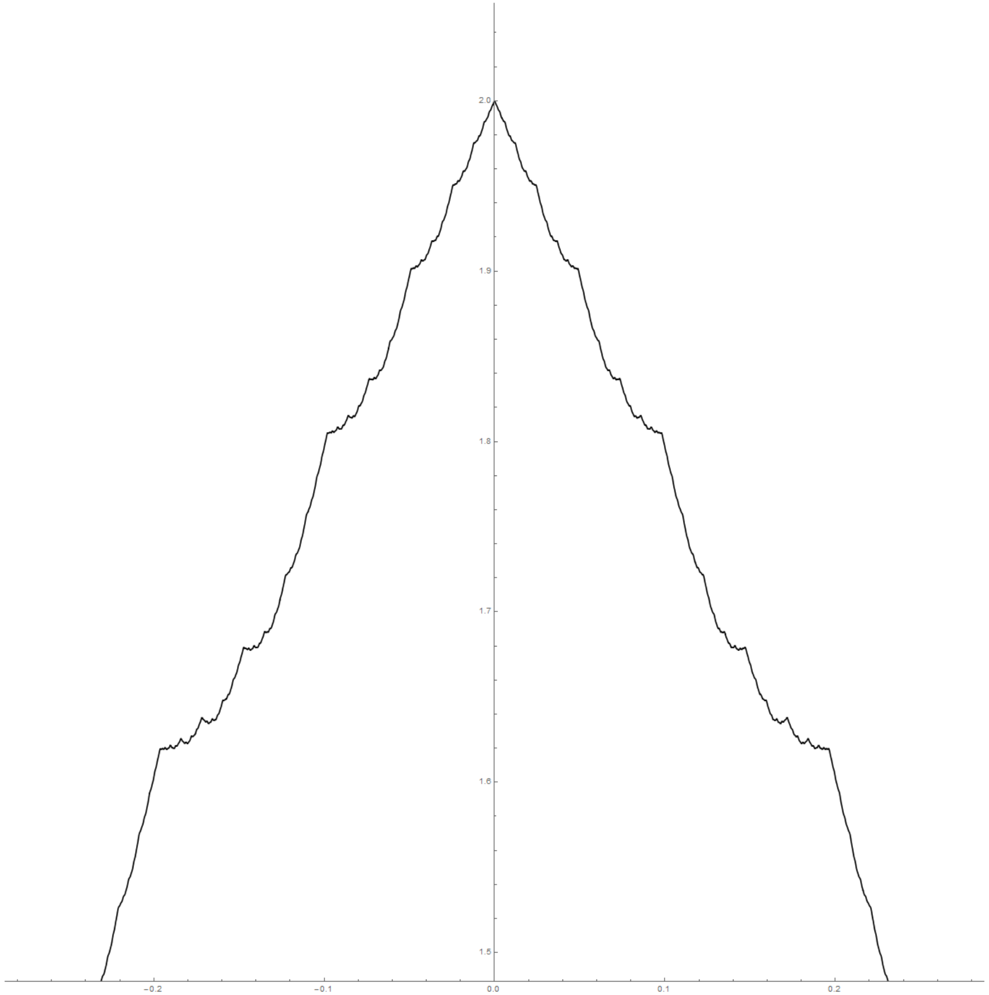

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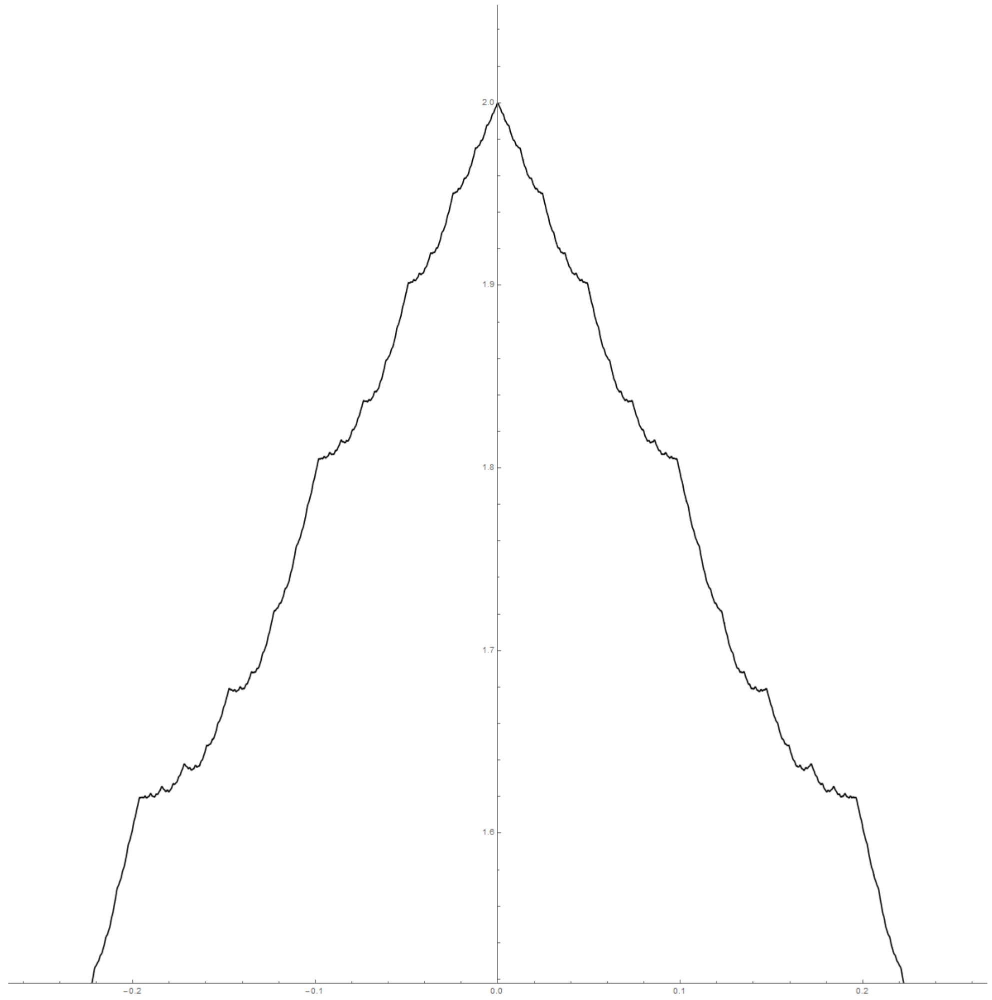

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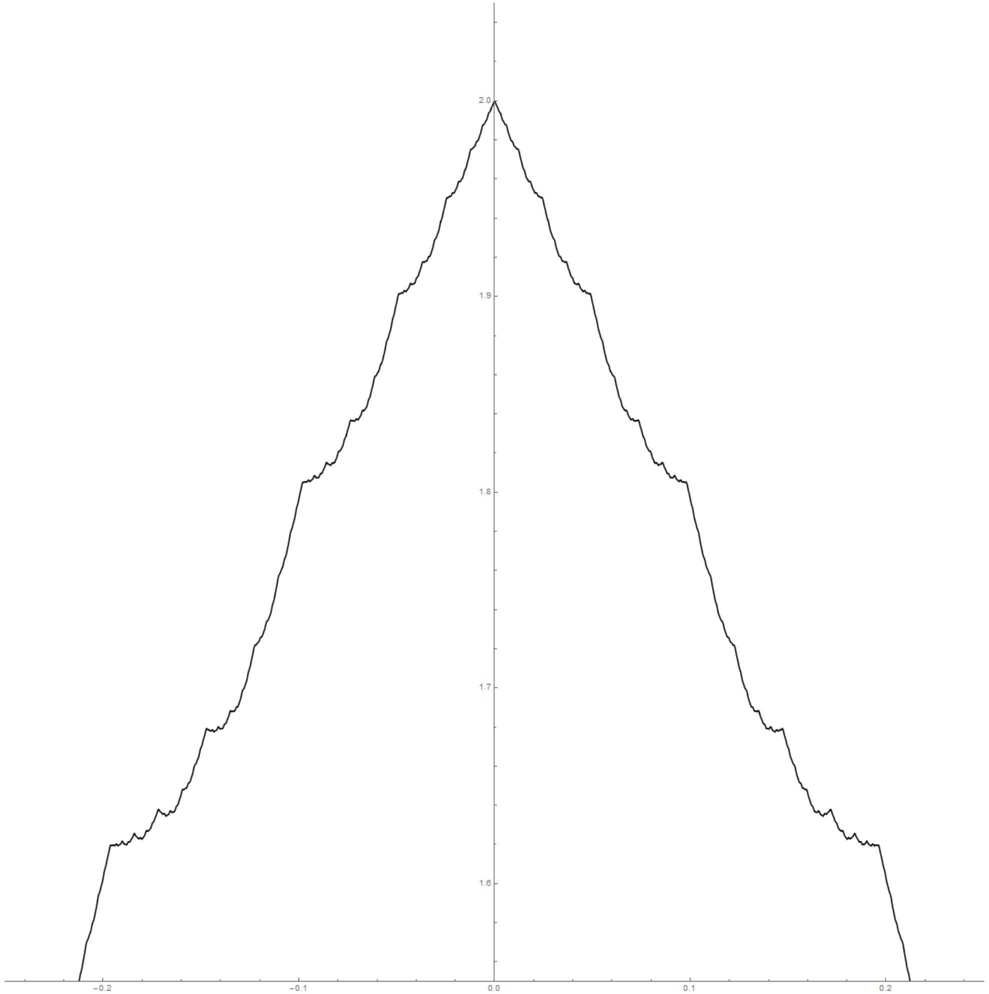

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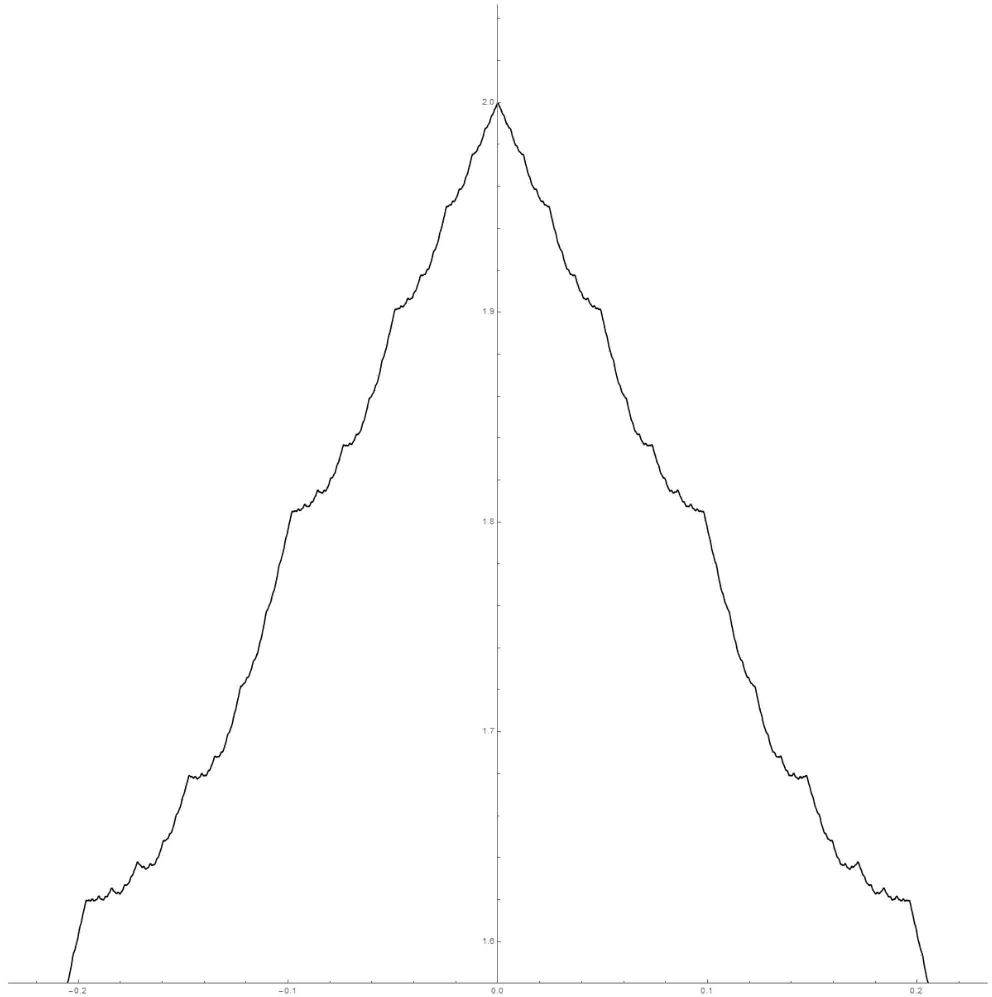

No matter how far you zoom in, the graph **never** settles down to a line!

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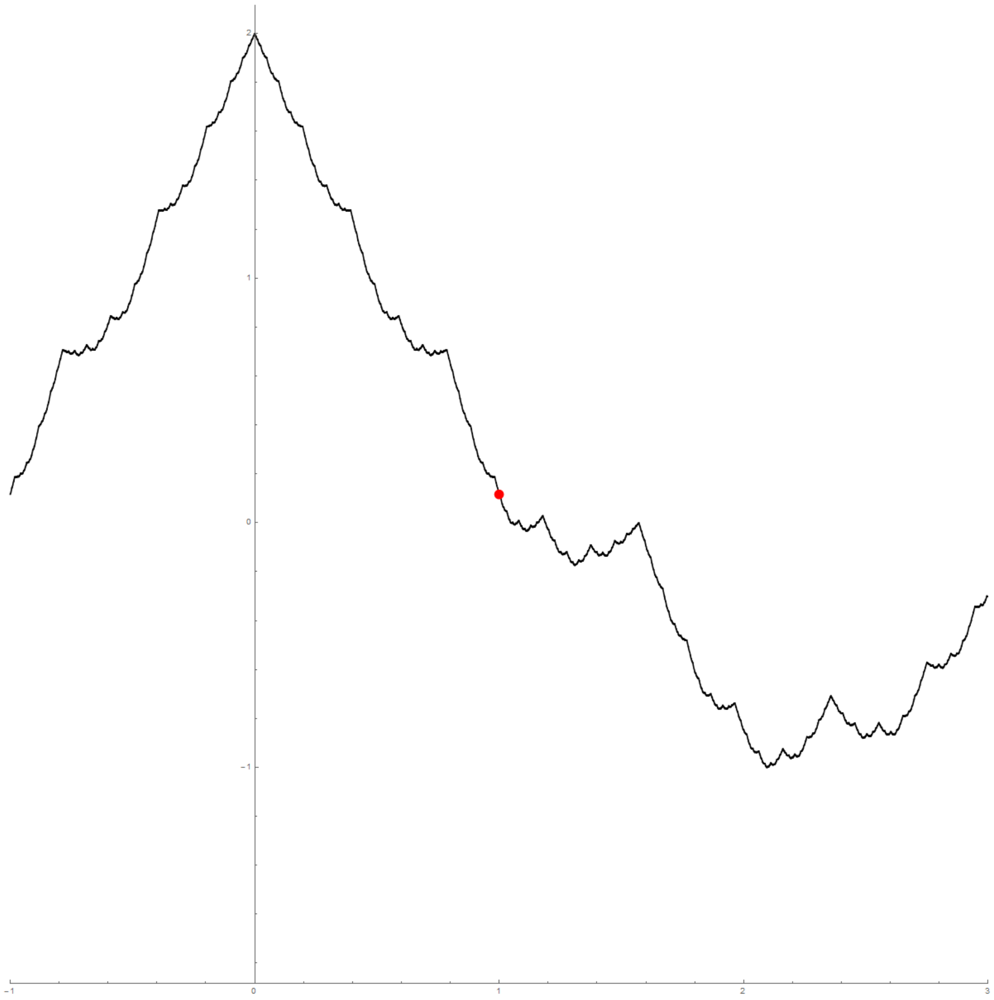

This time we zoom in to $(1, 0.117078)$ on the graph (when $x = 1$, $y \approx 0.117078$.)

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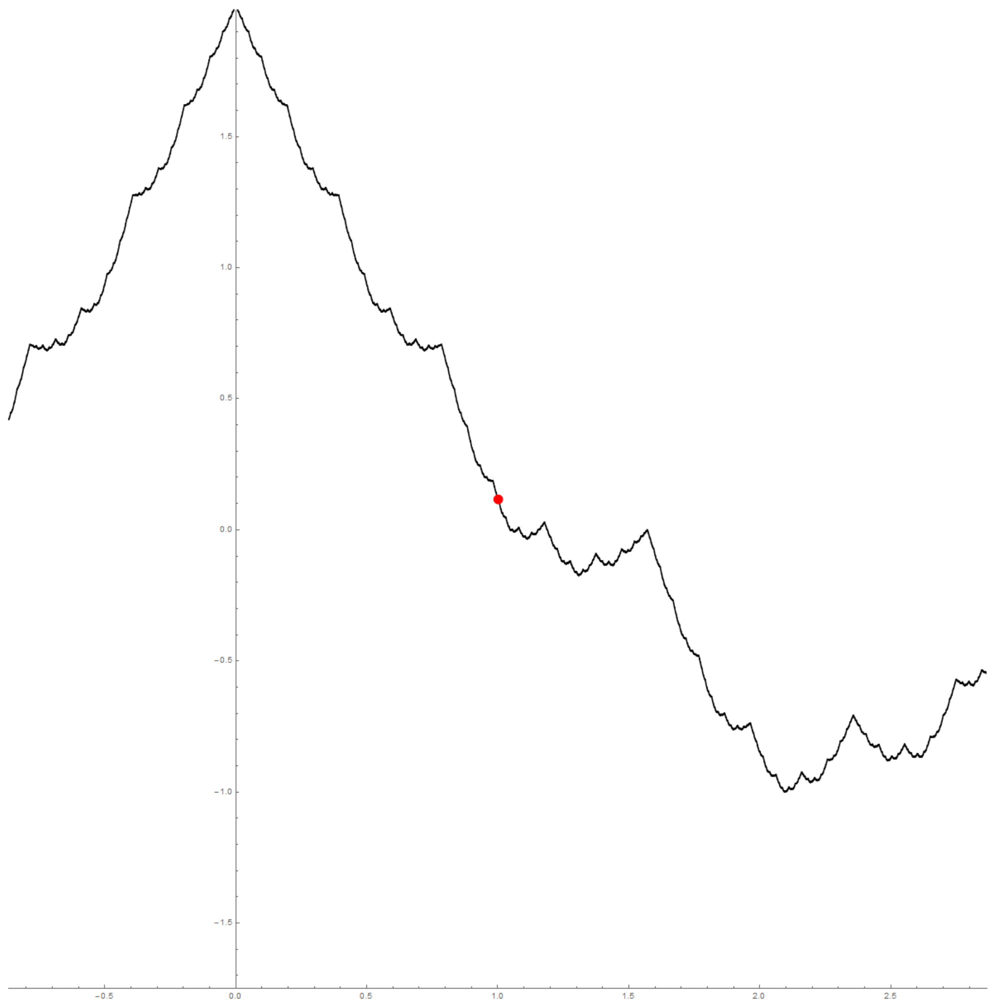

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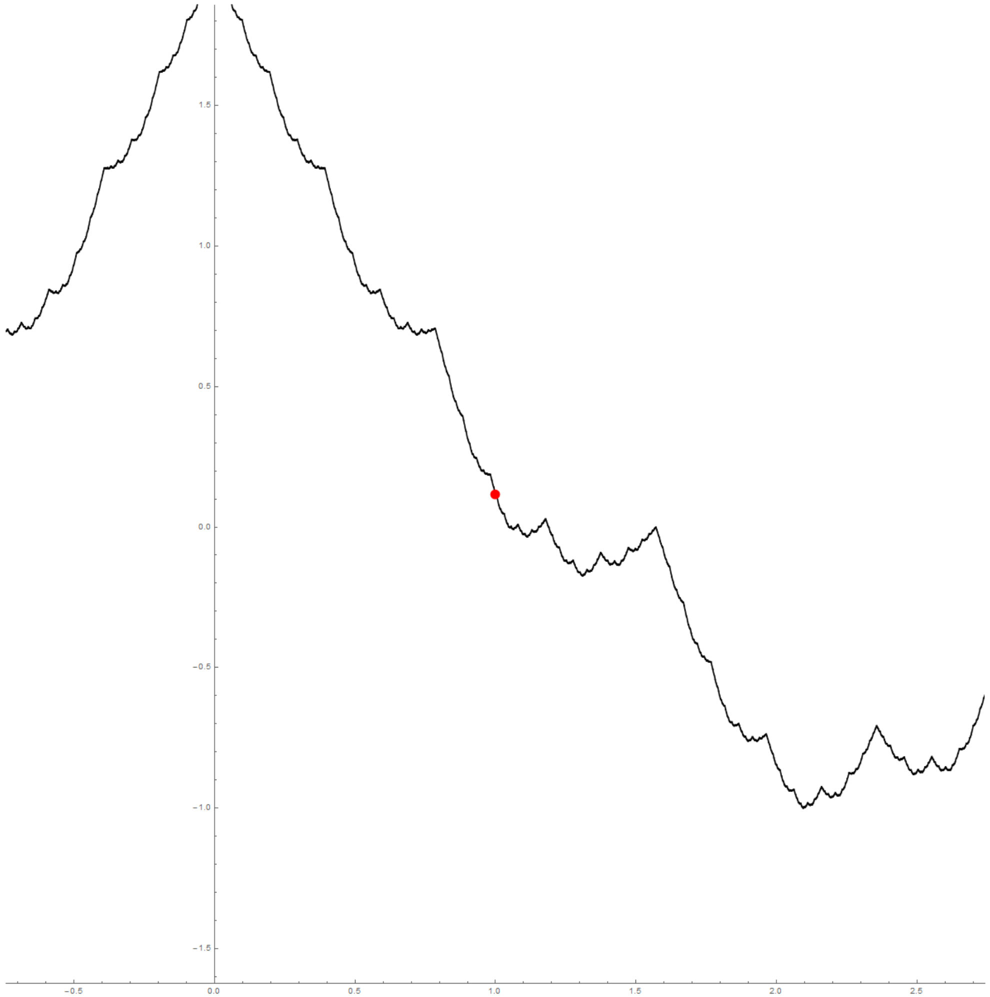

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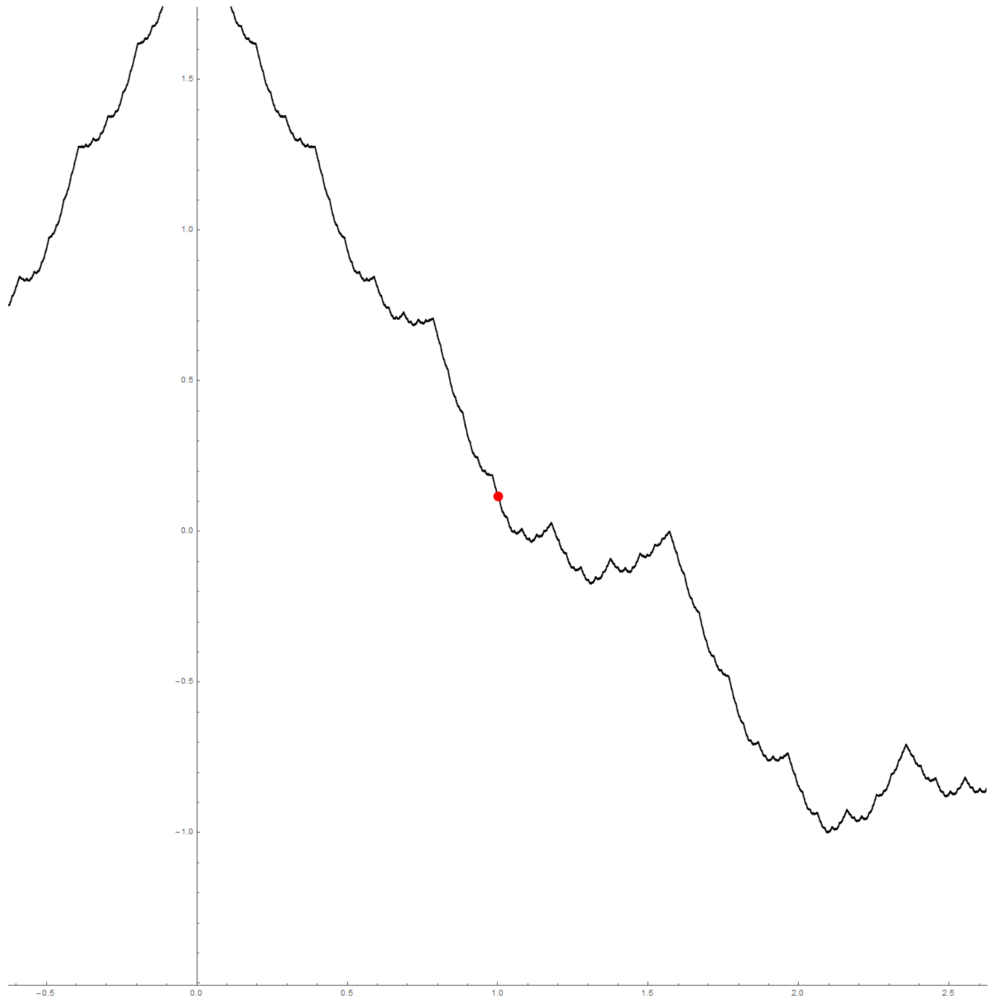

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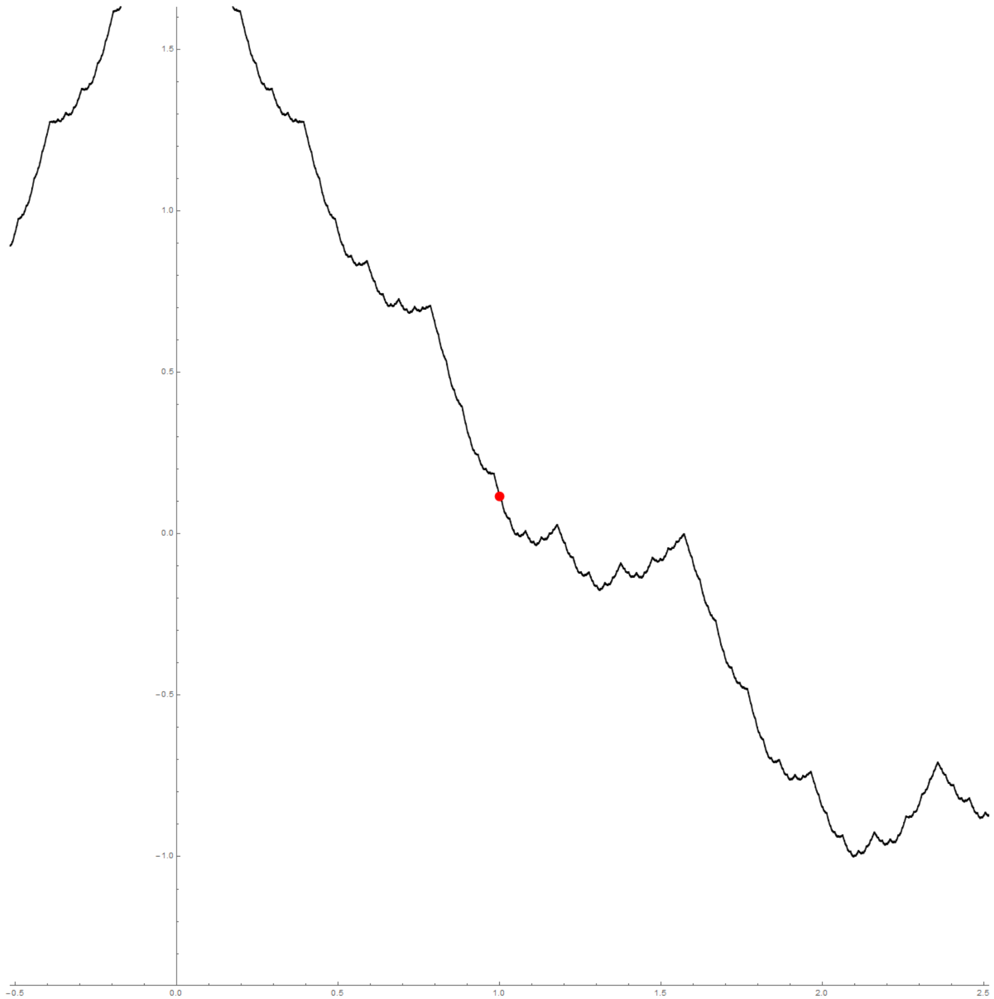

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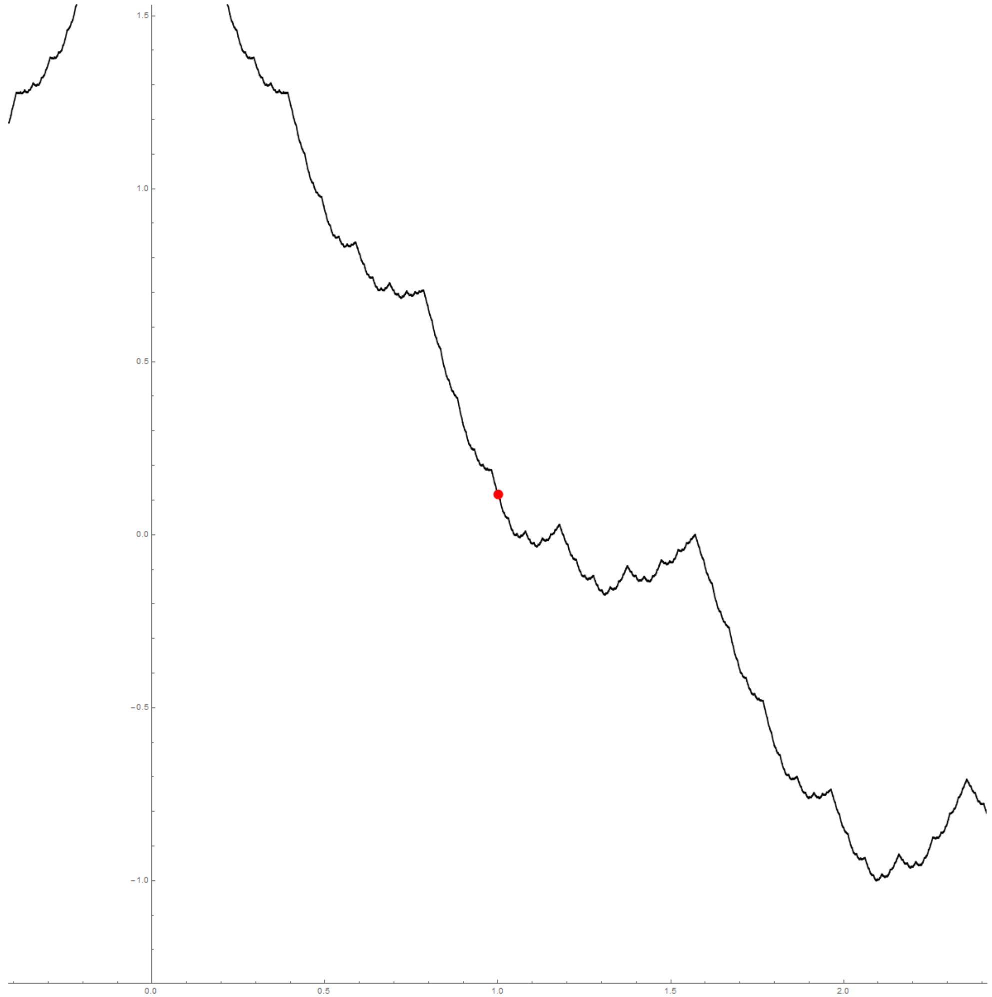

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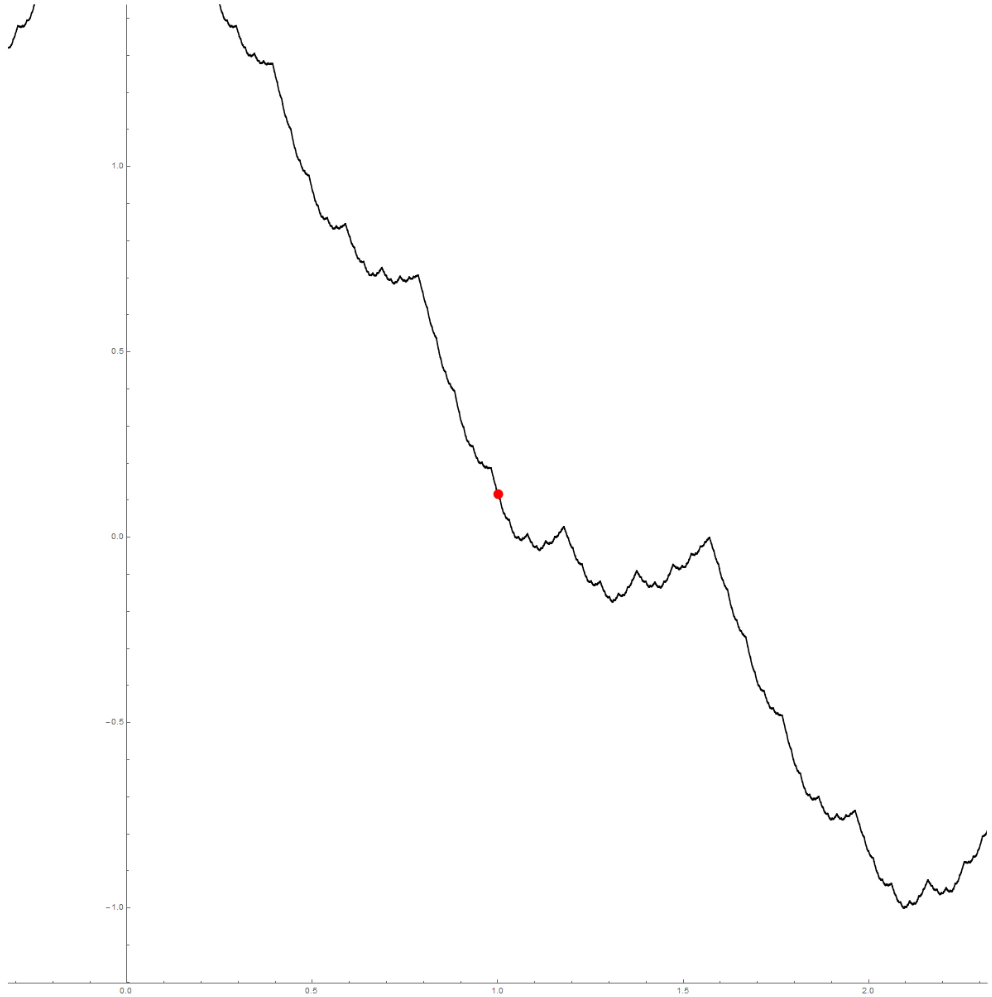

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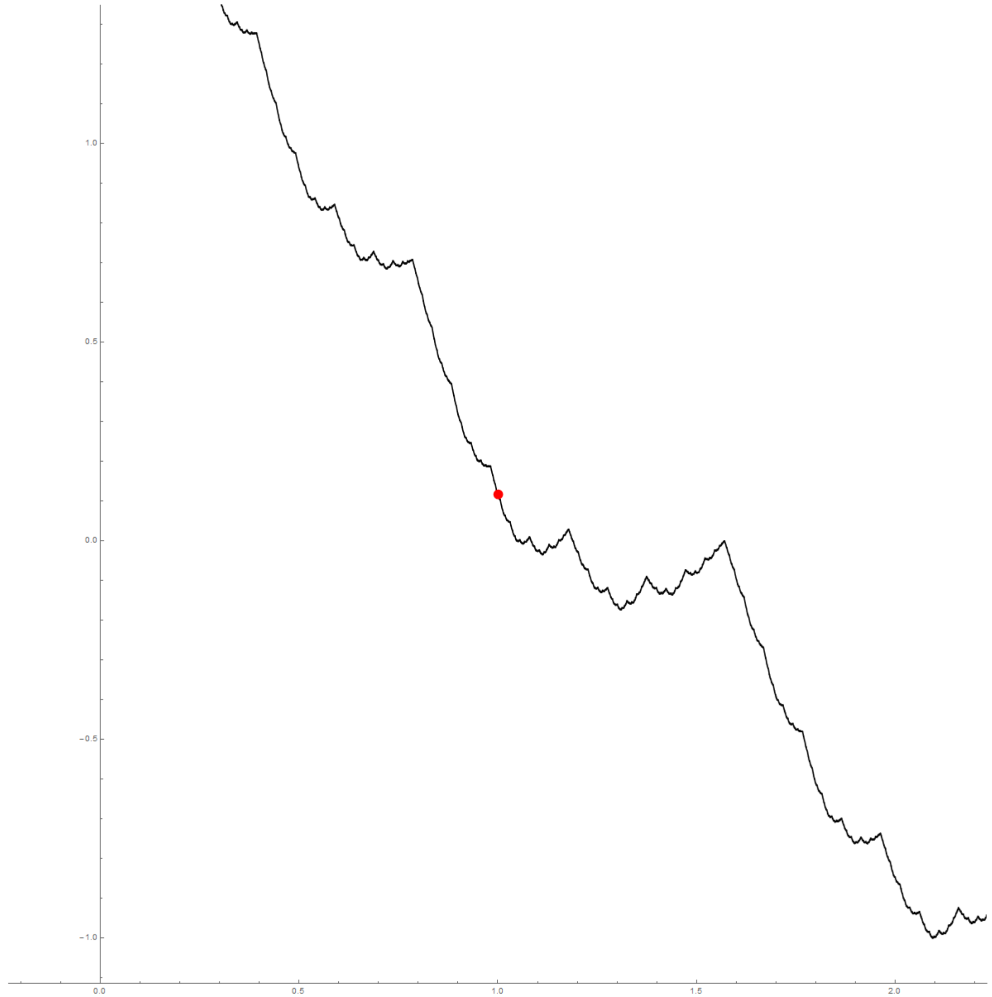

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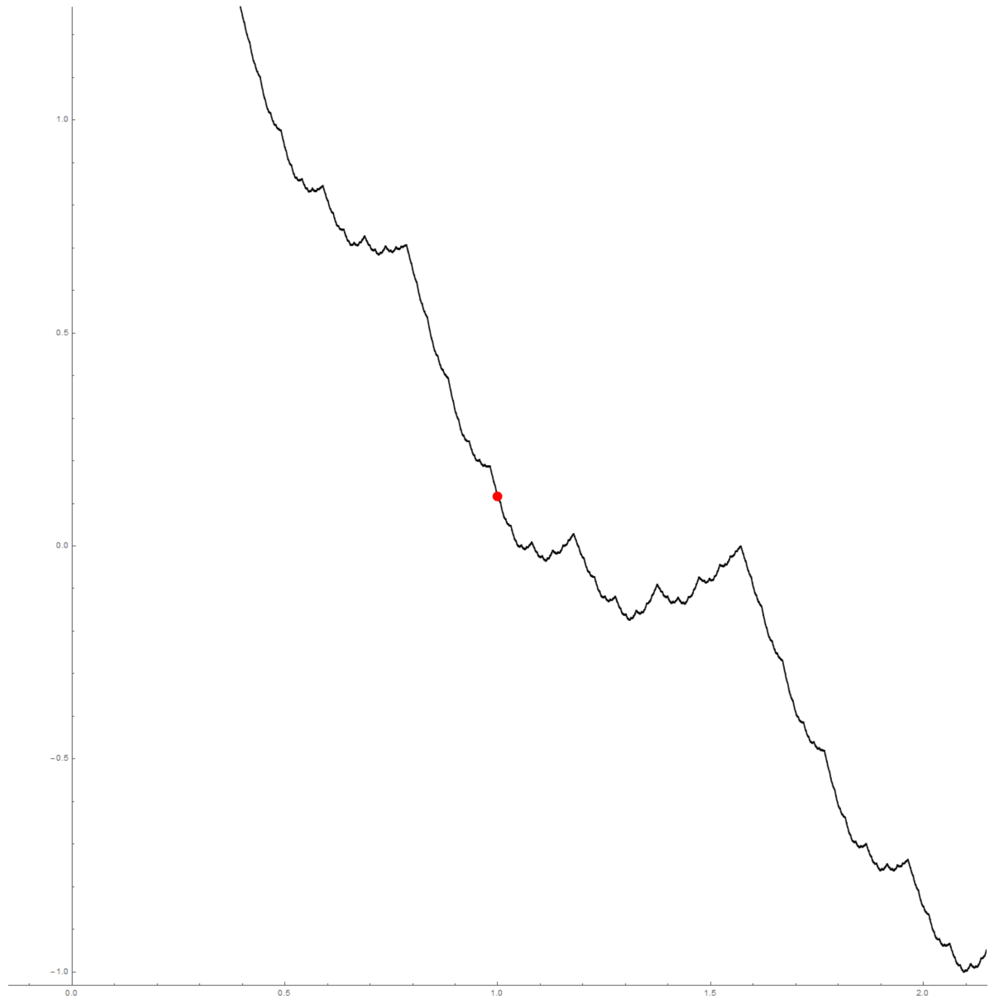

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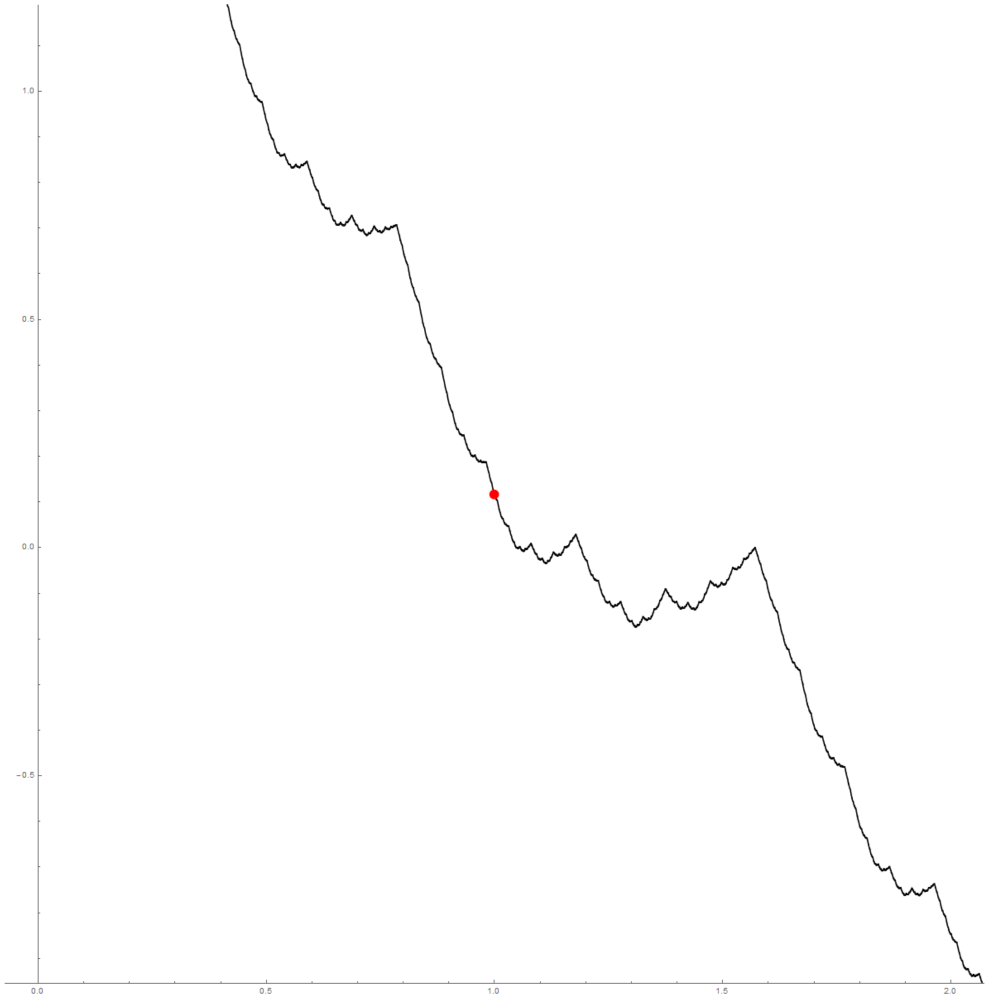

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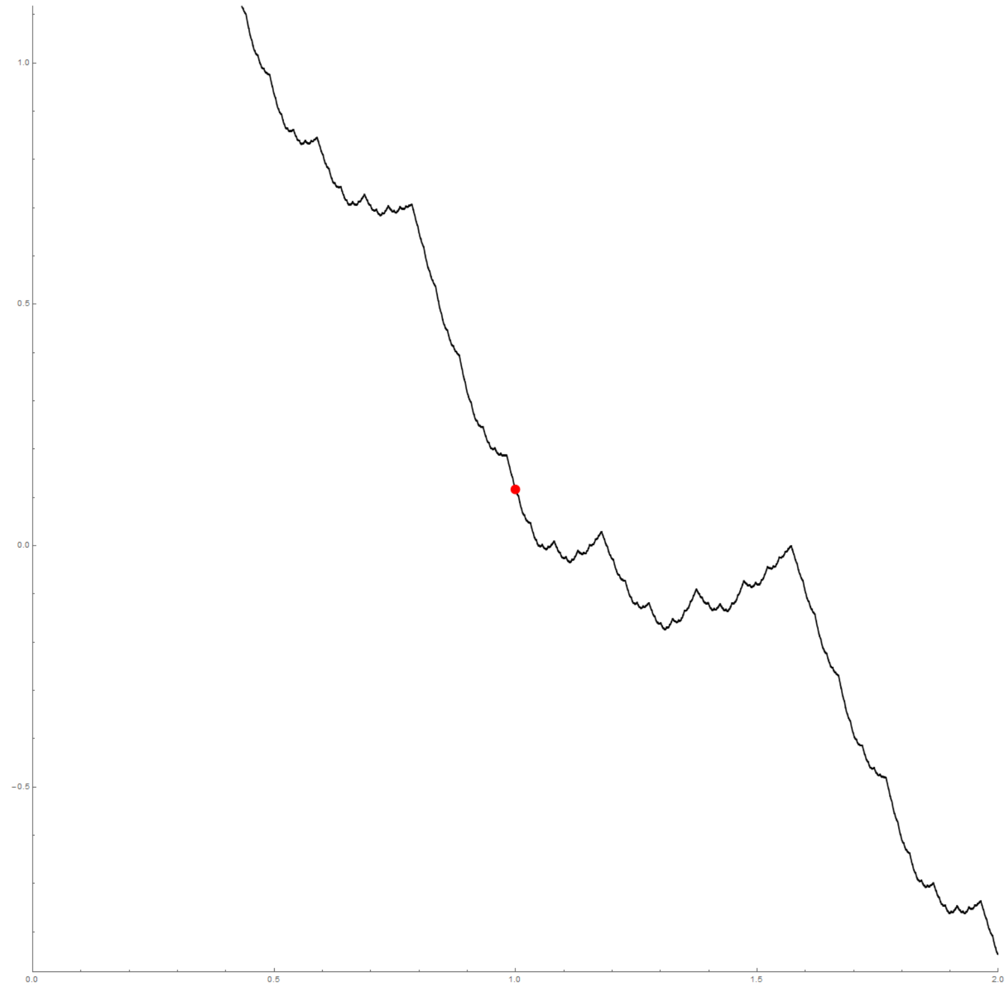

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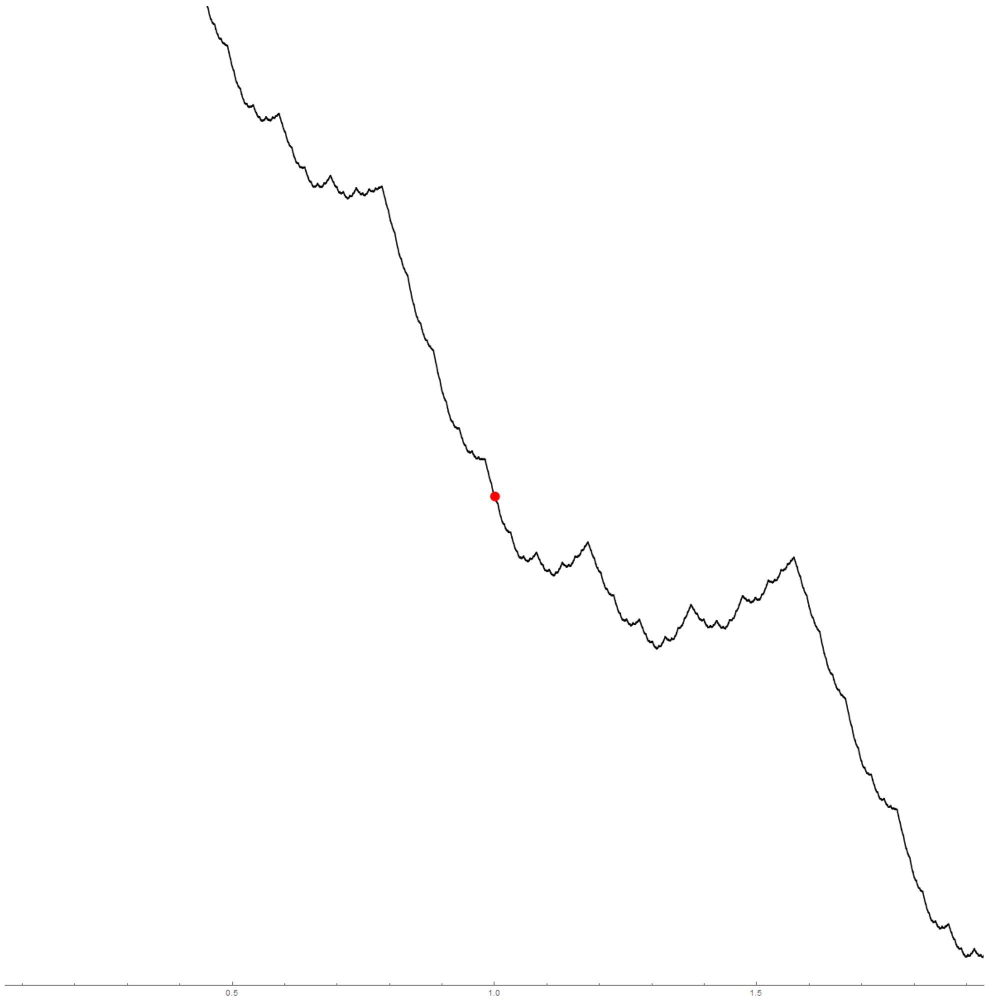

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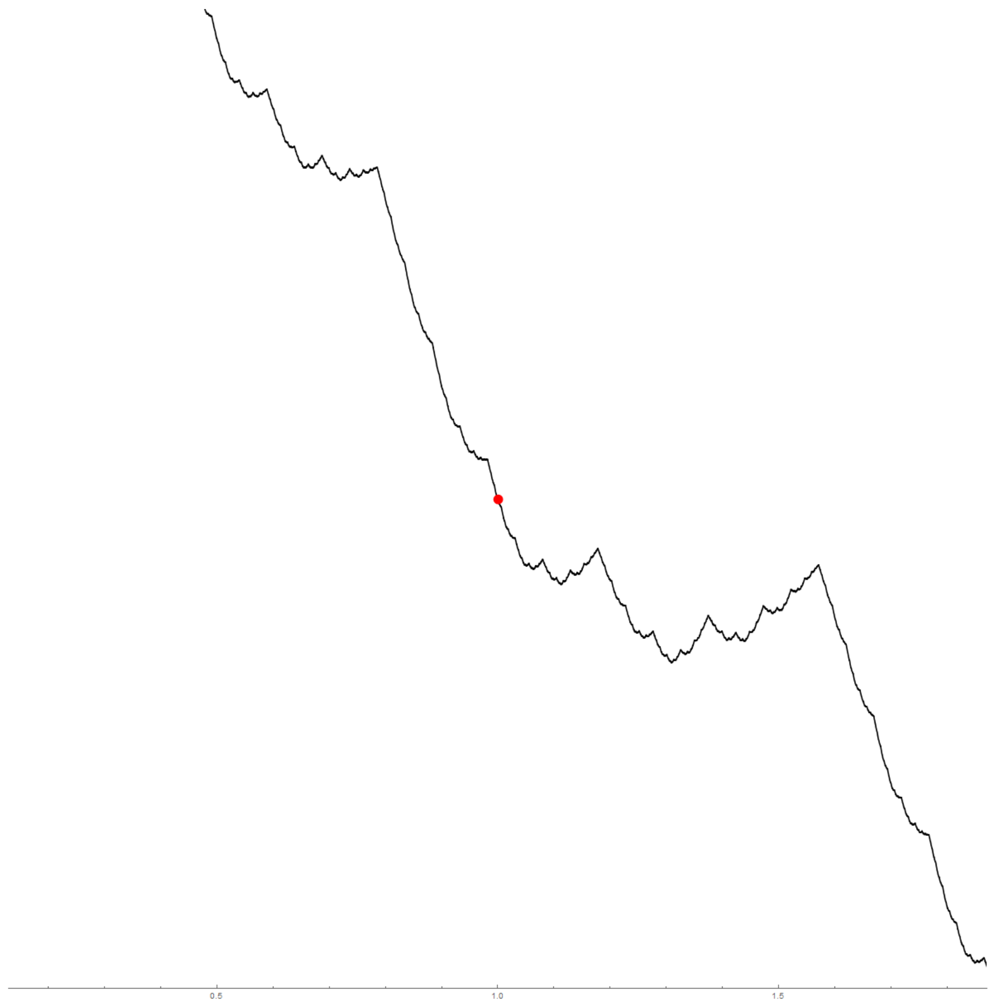

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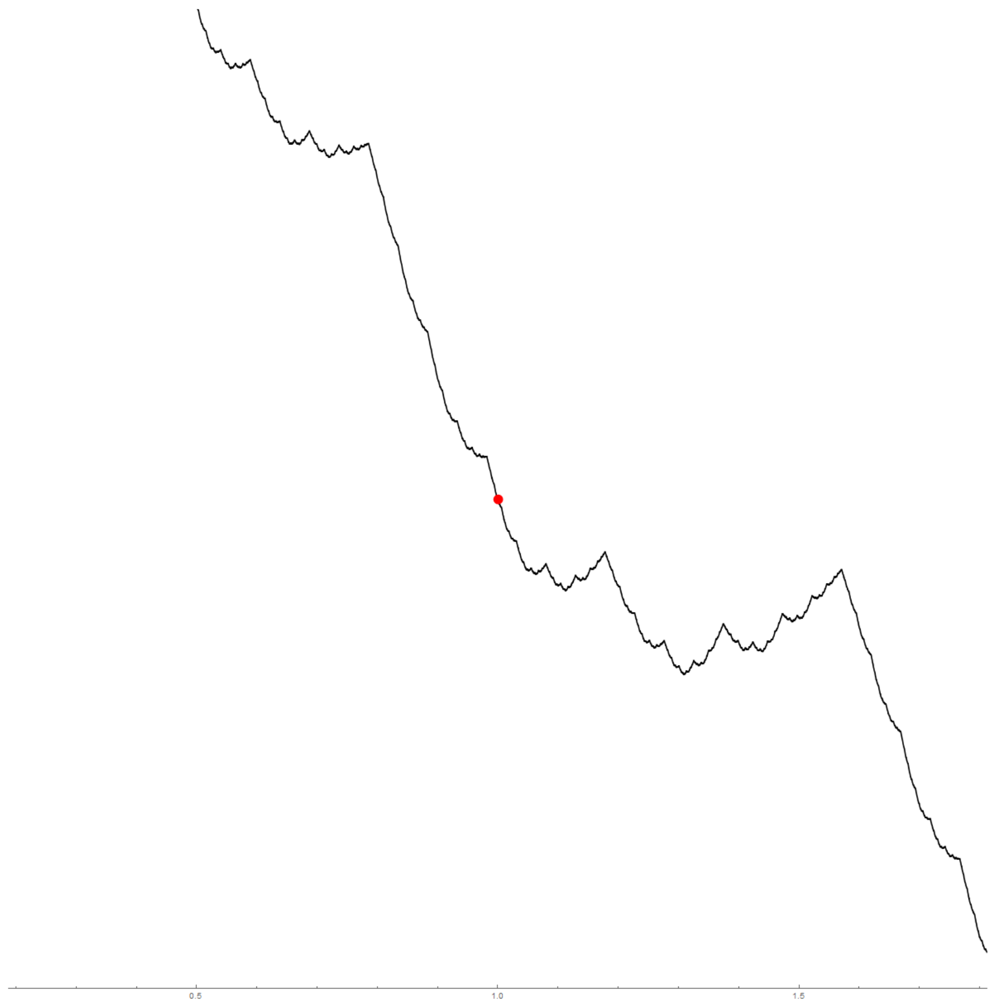

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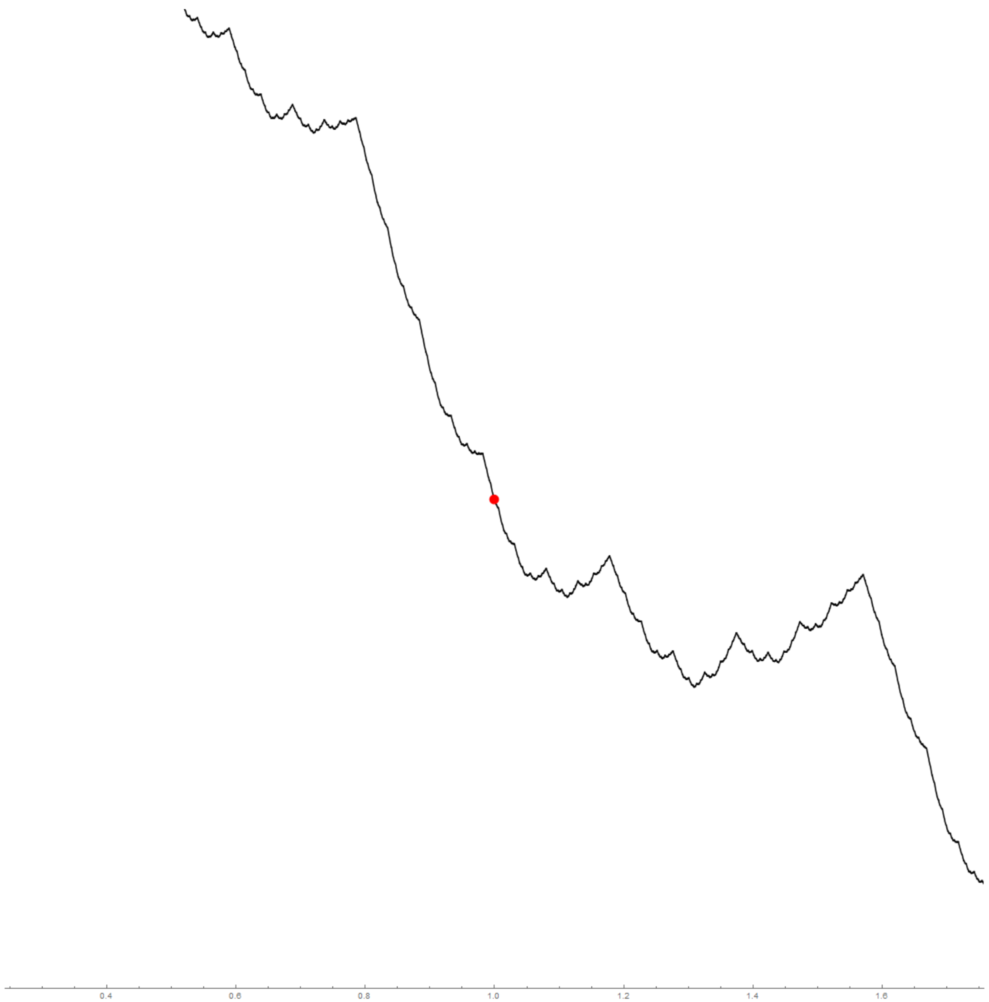

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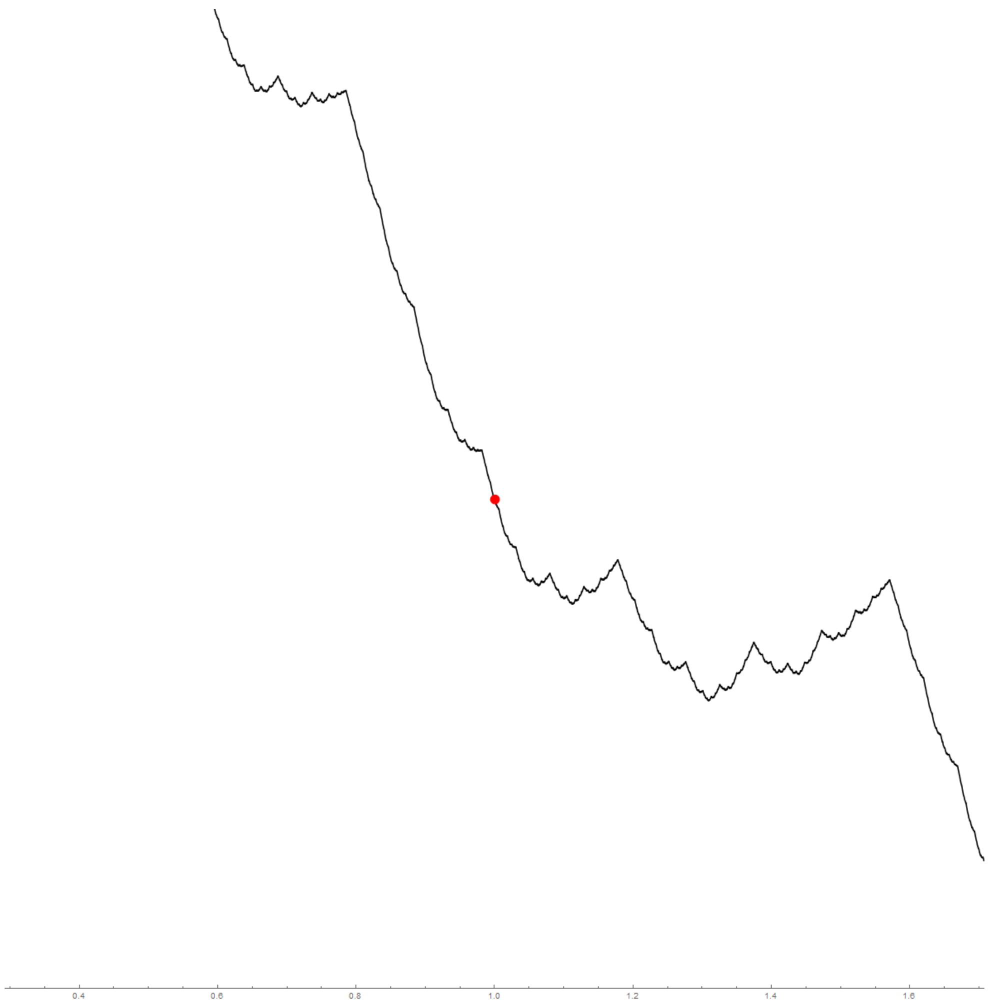

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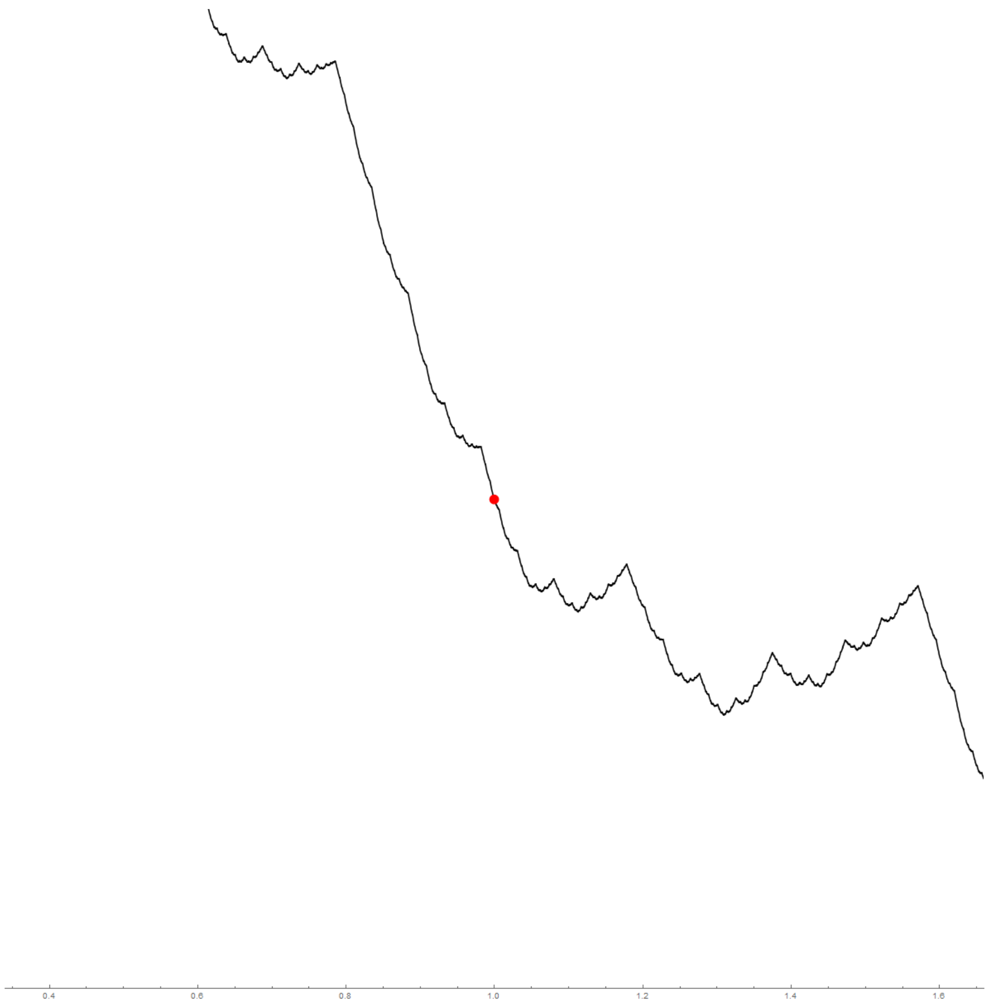

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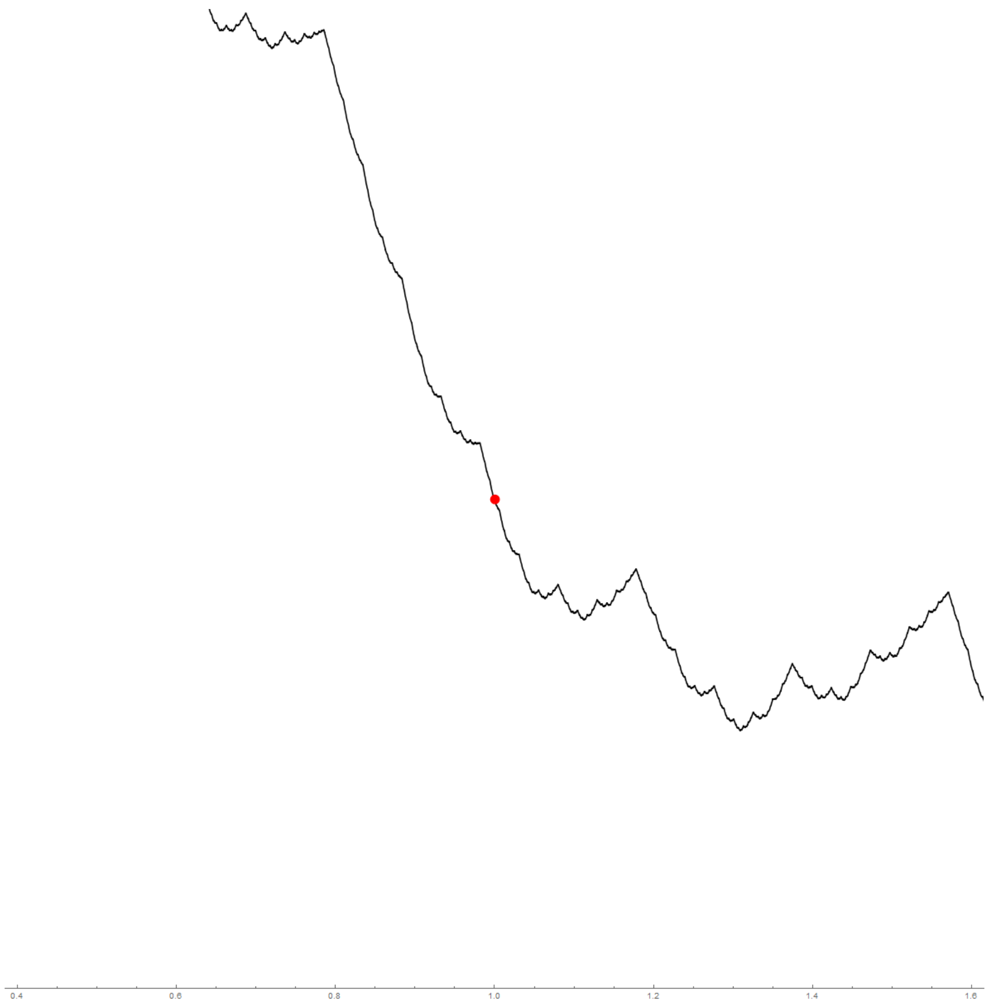

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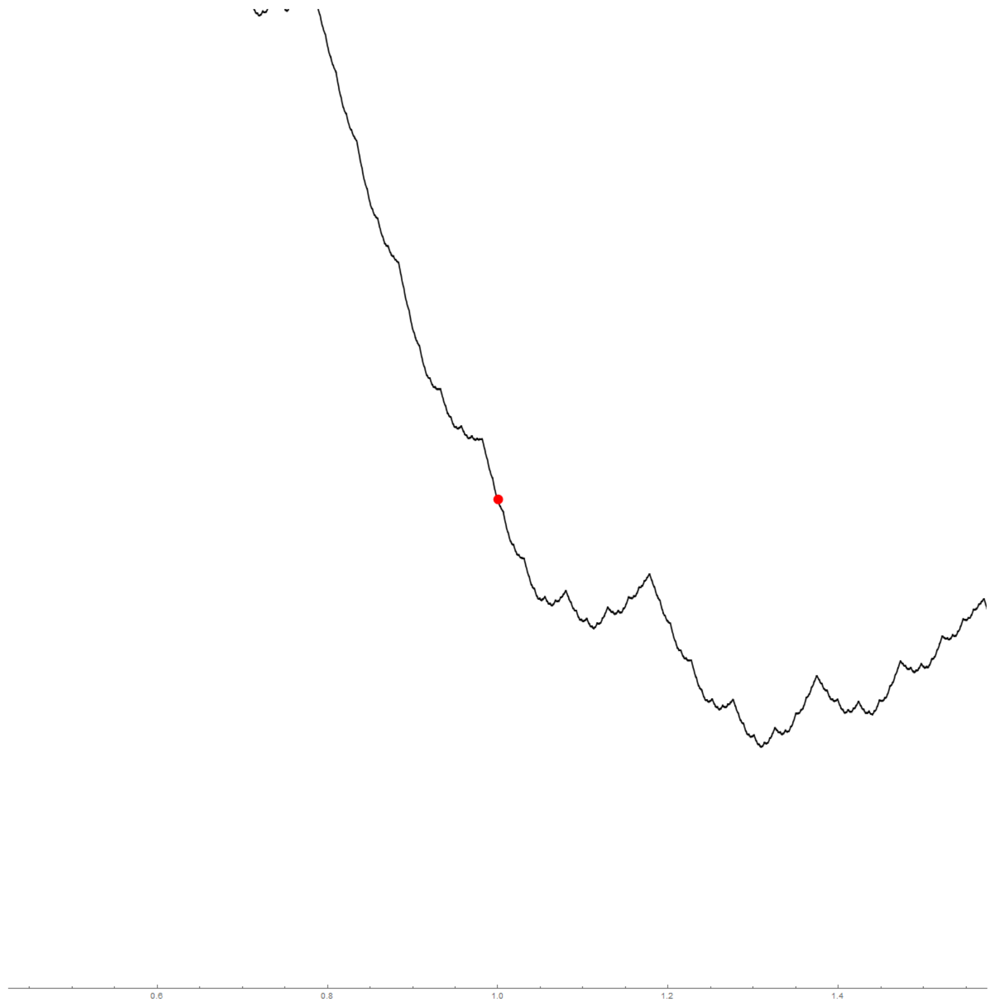

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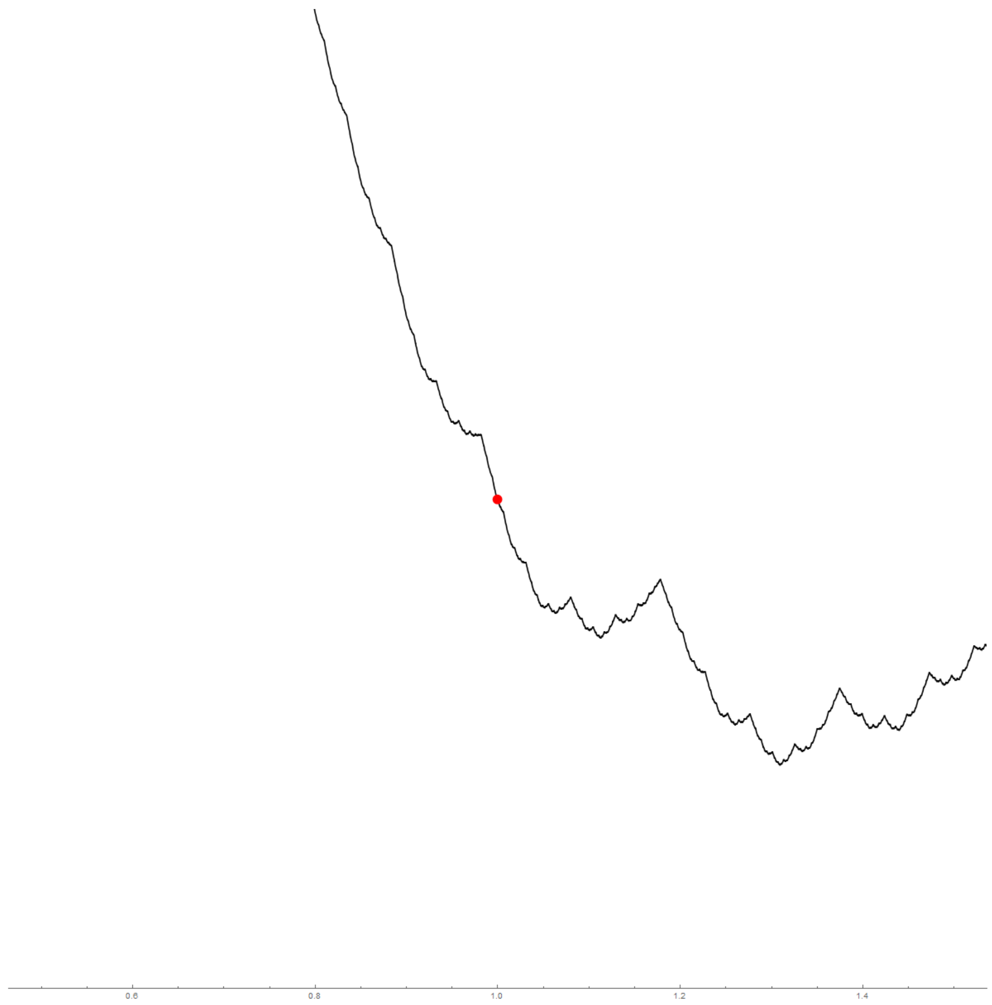

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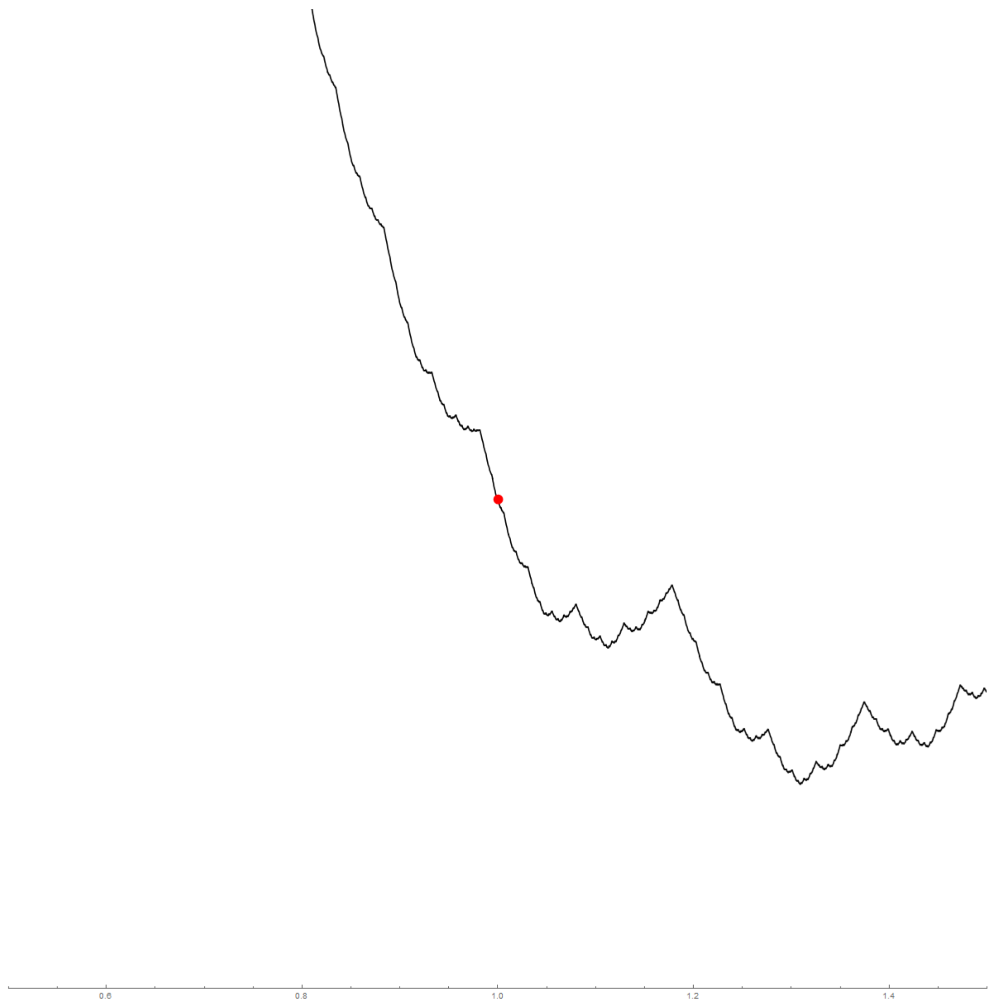

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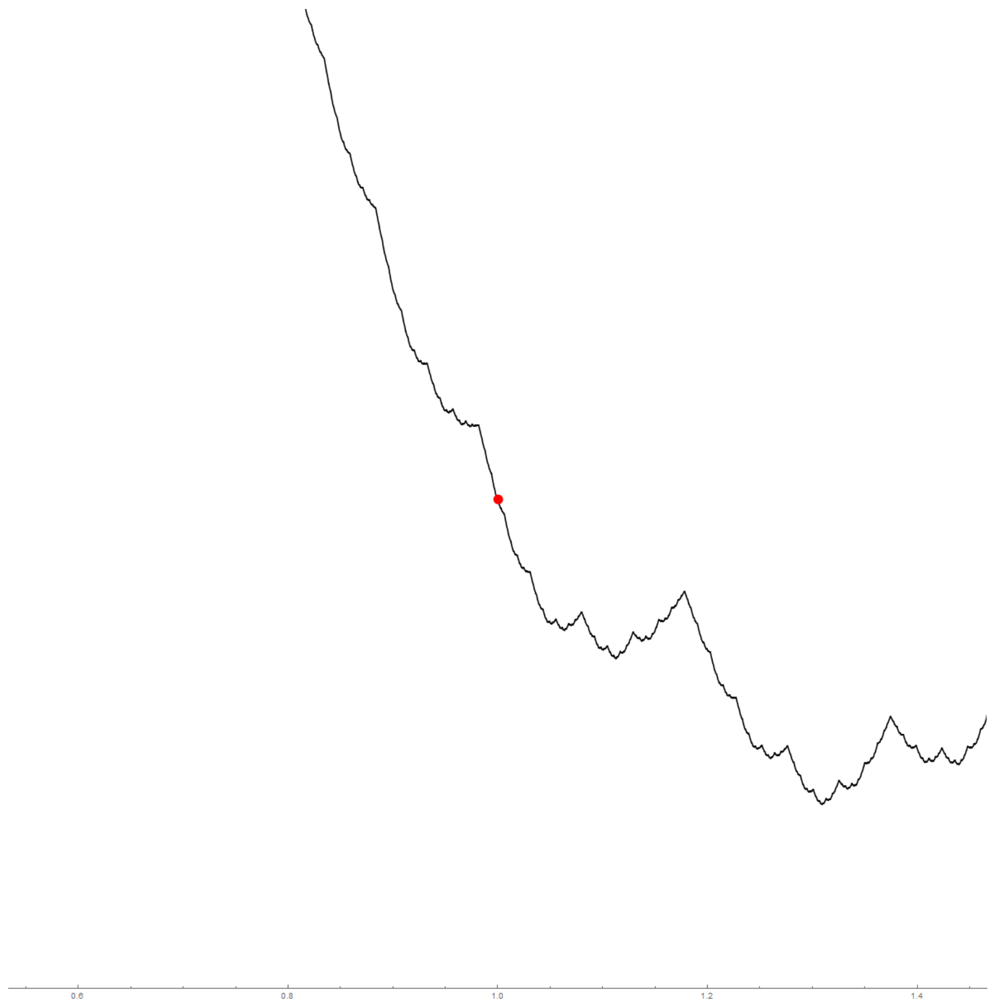

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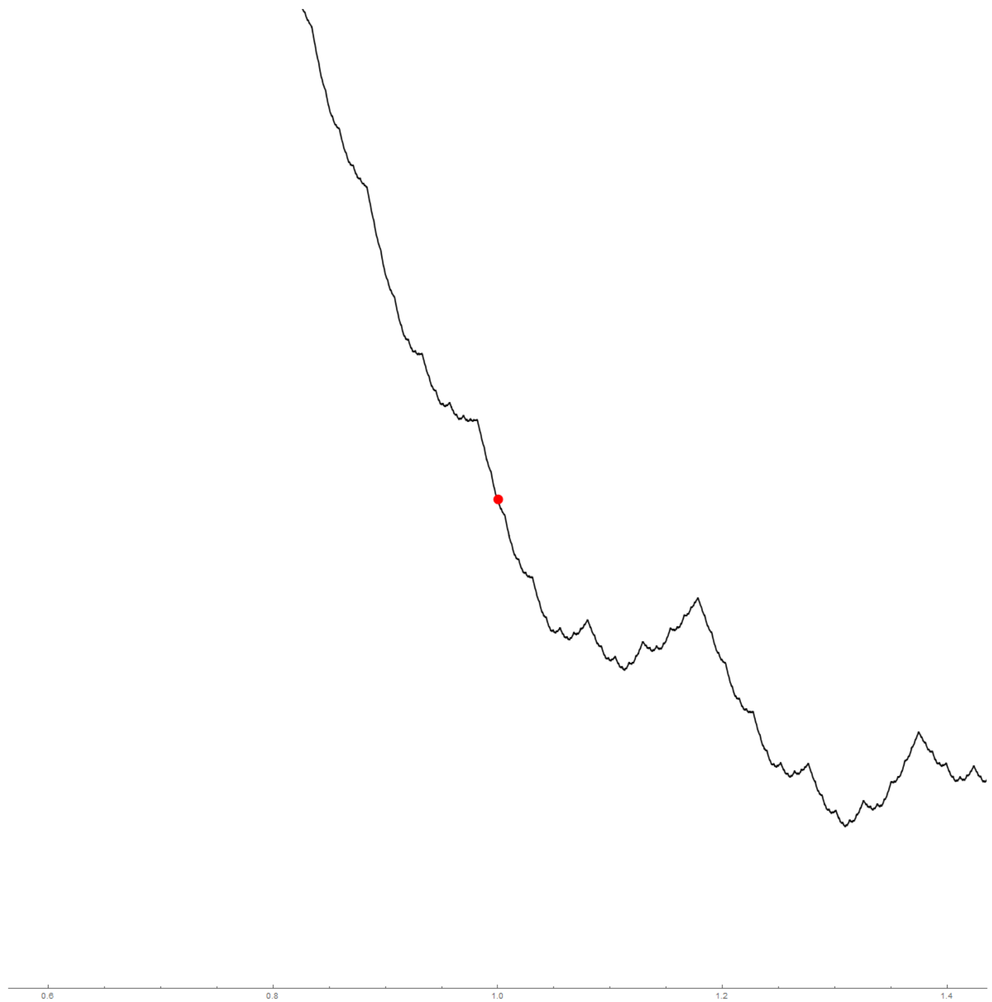

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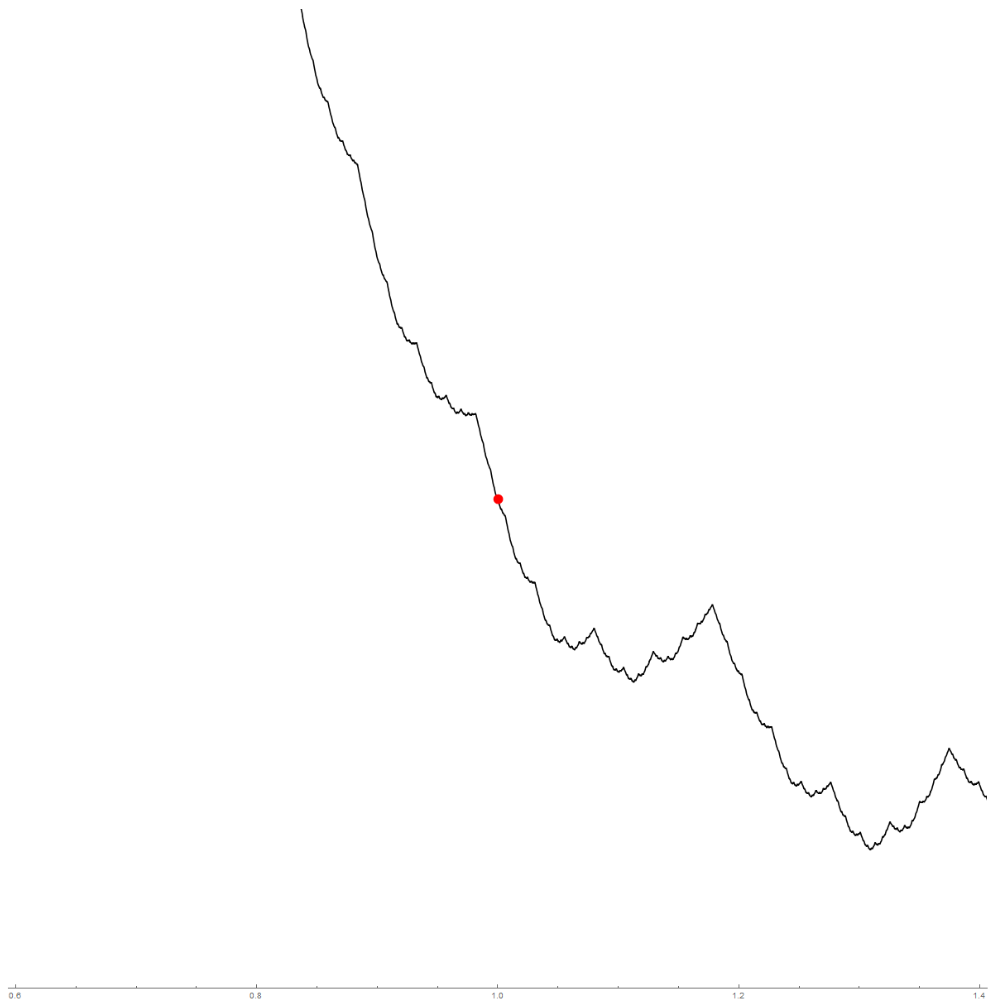

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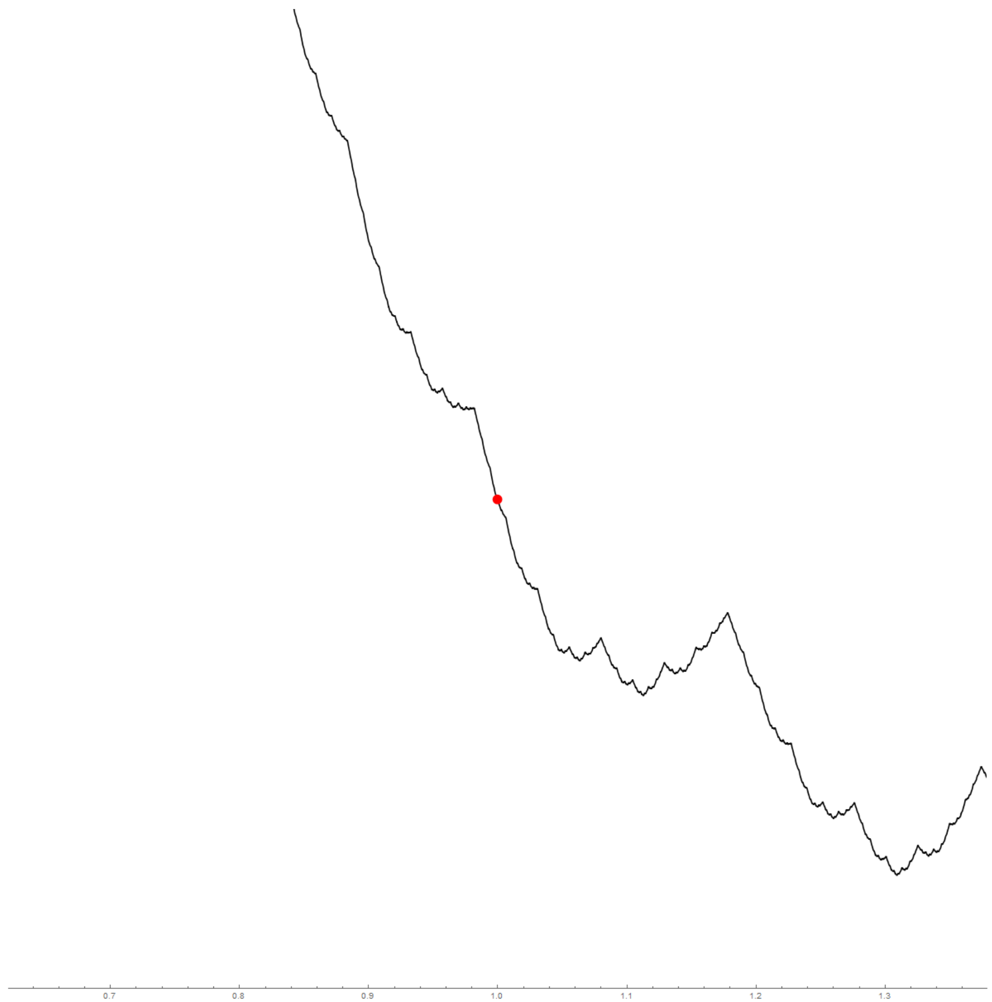

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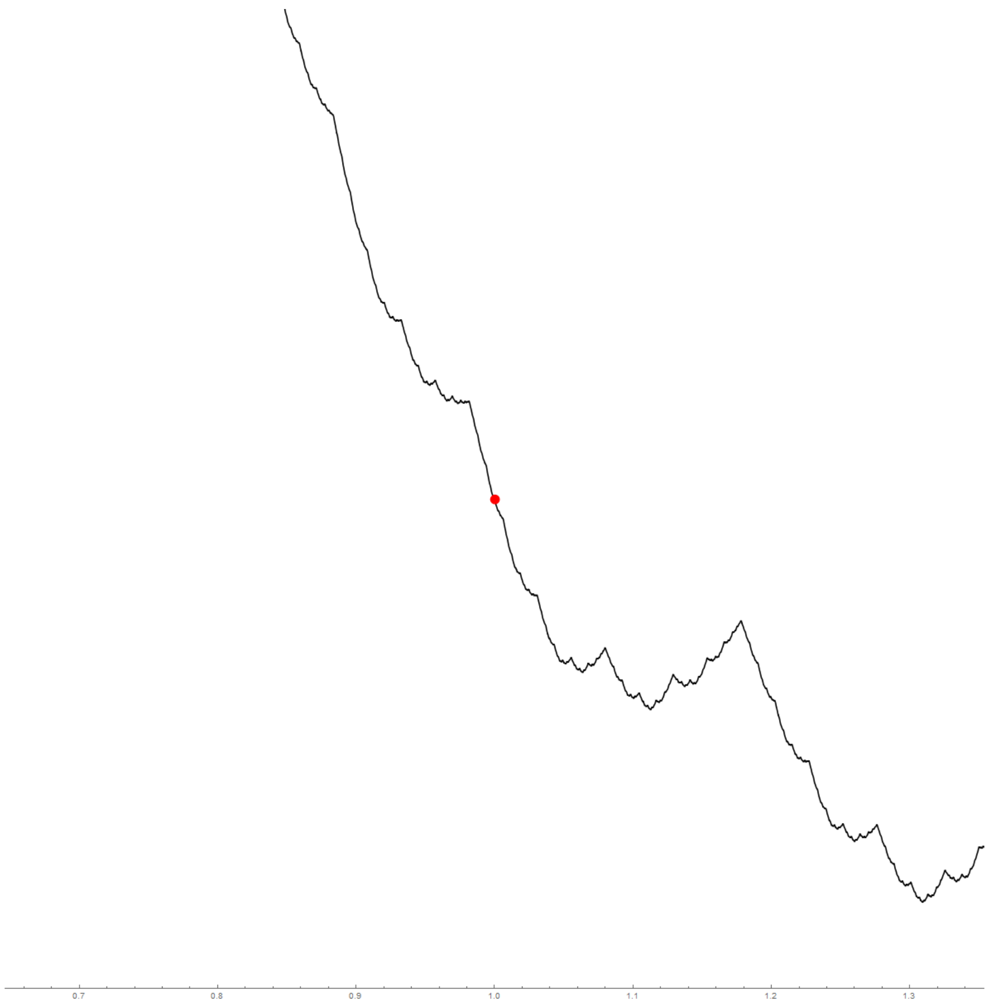

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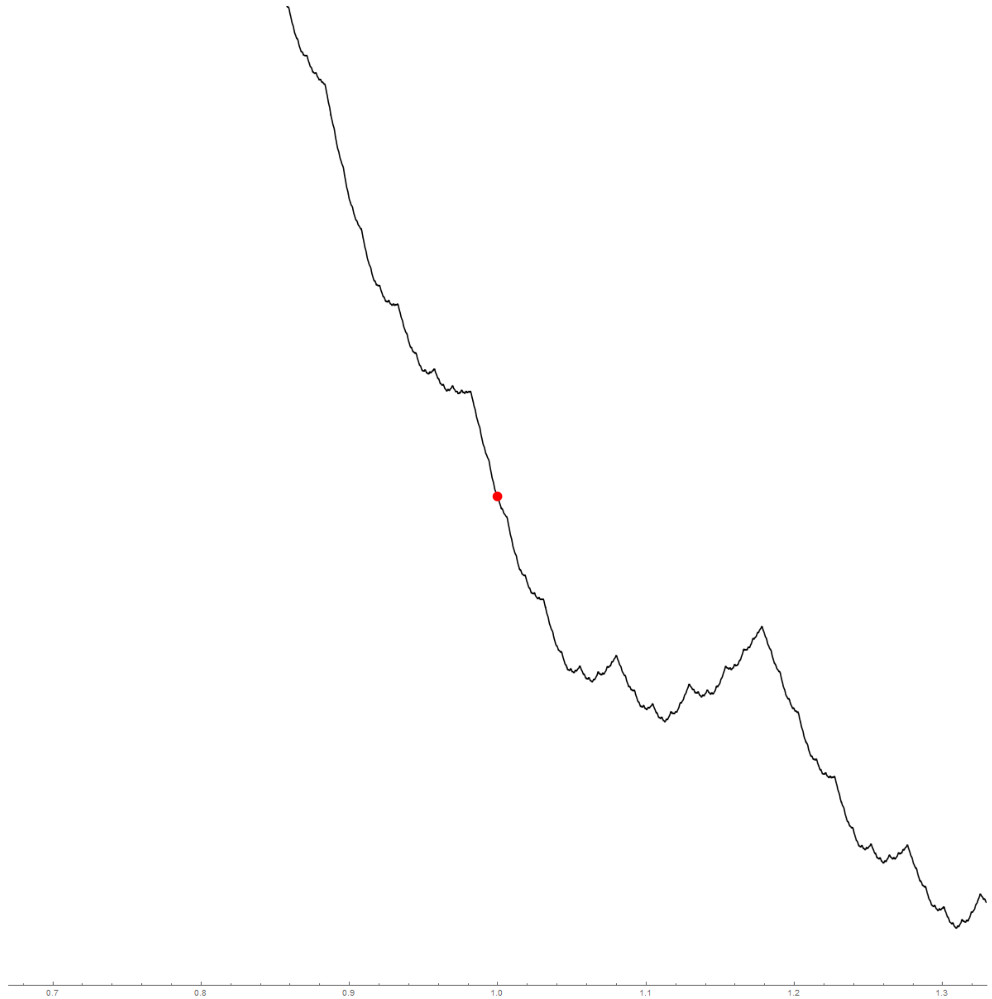

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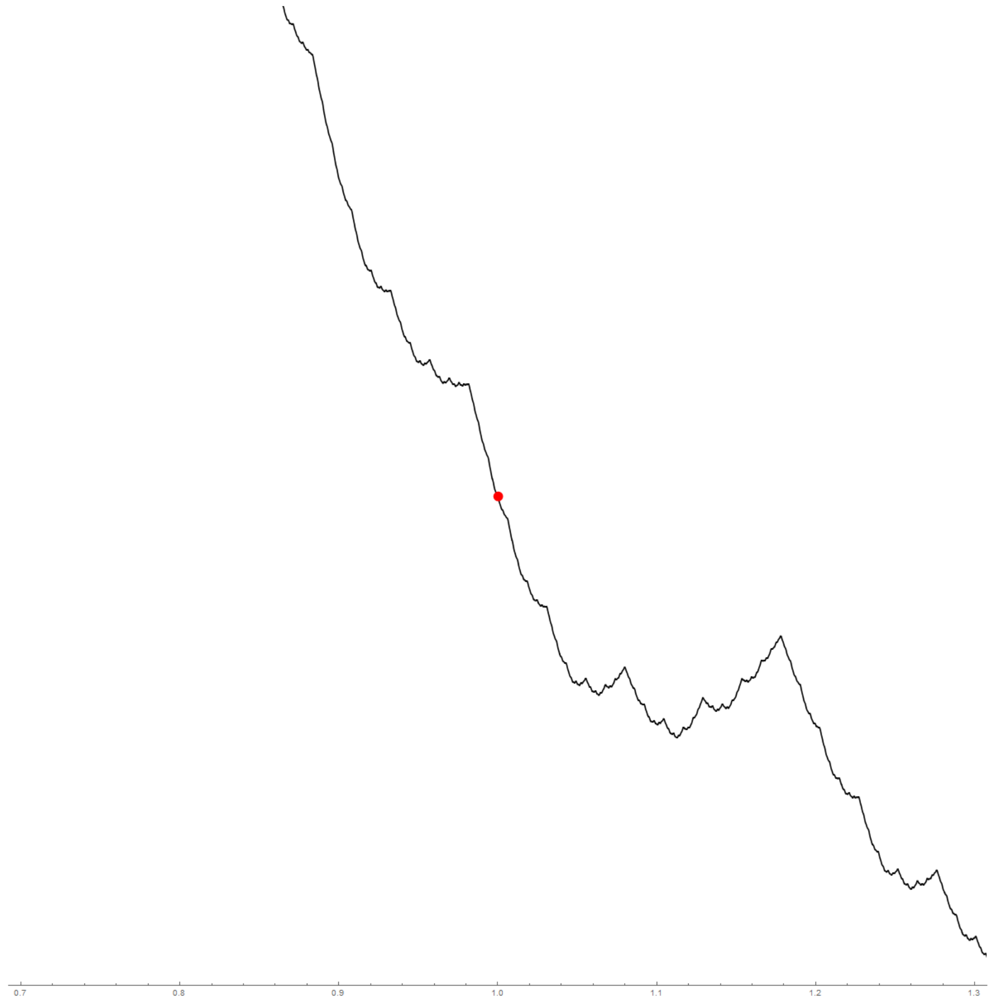

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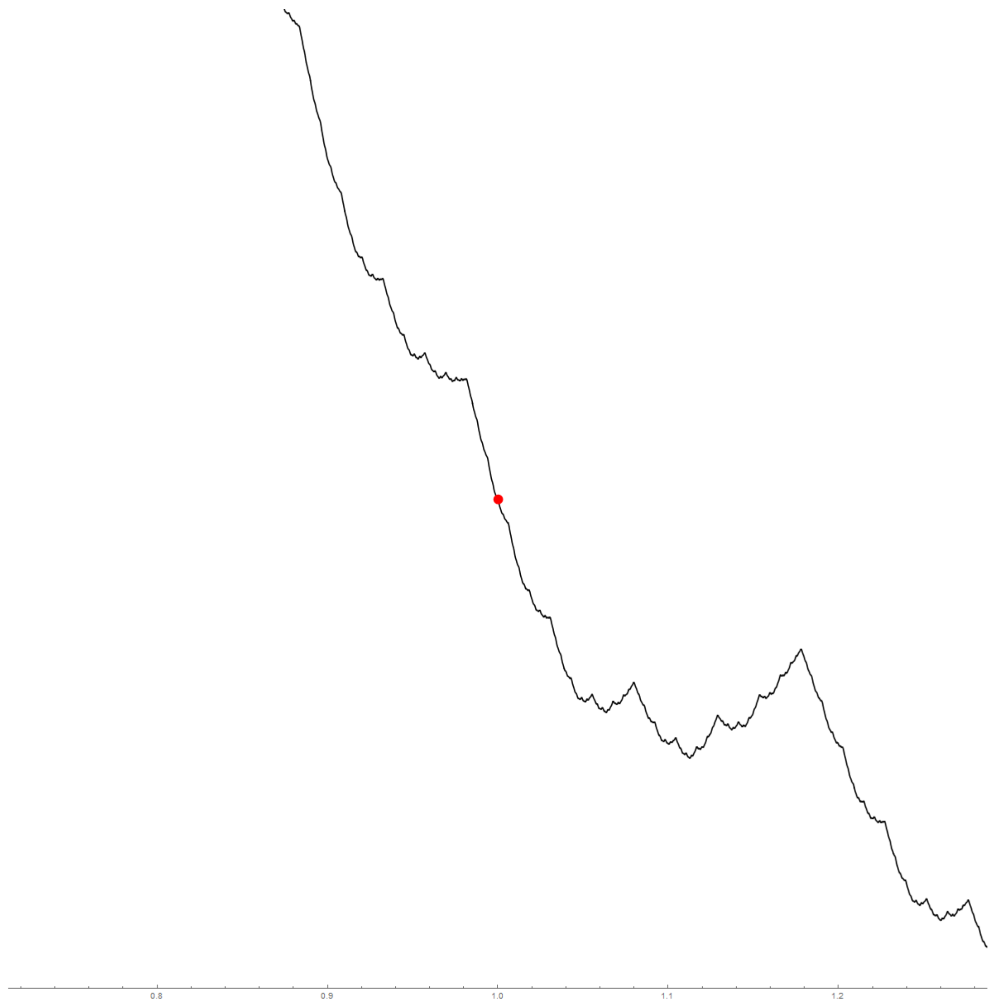

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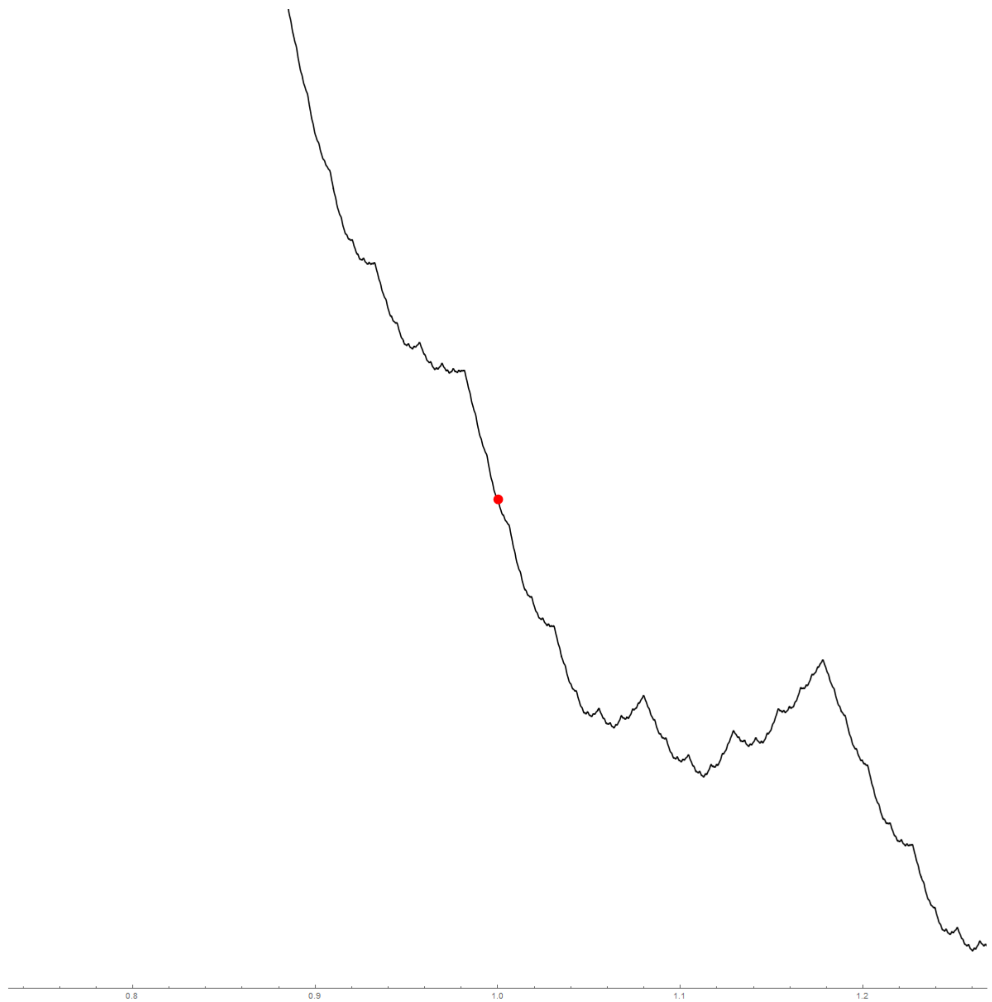

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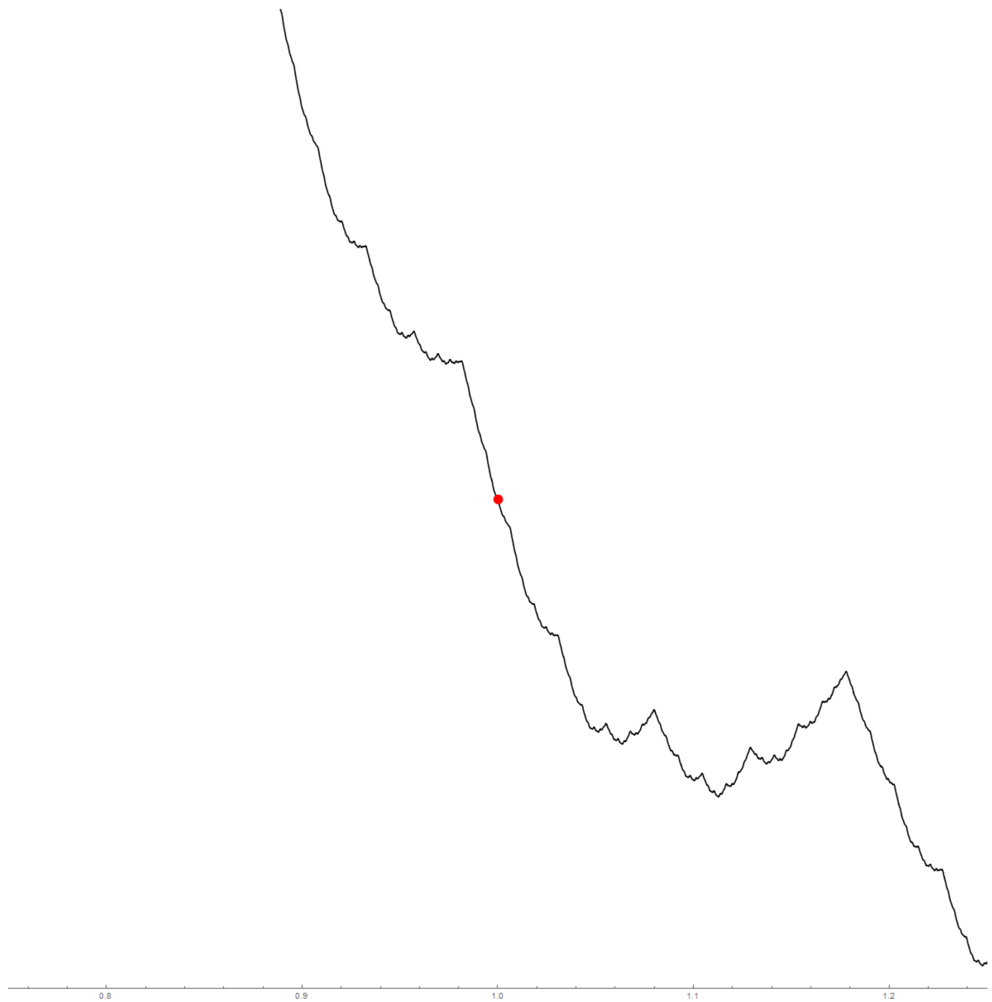

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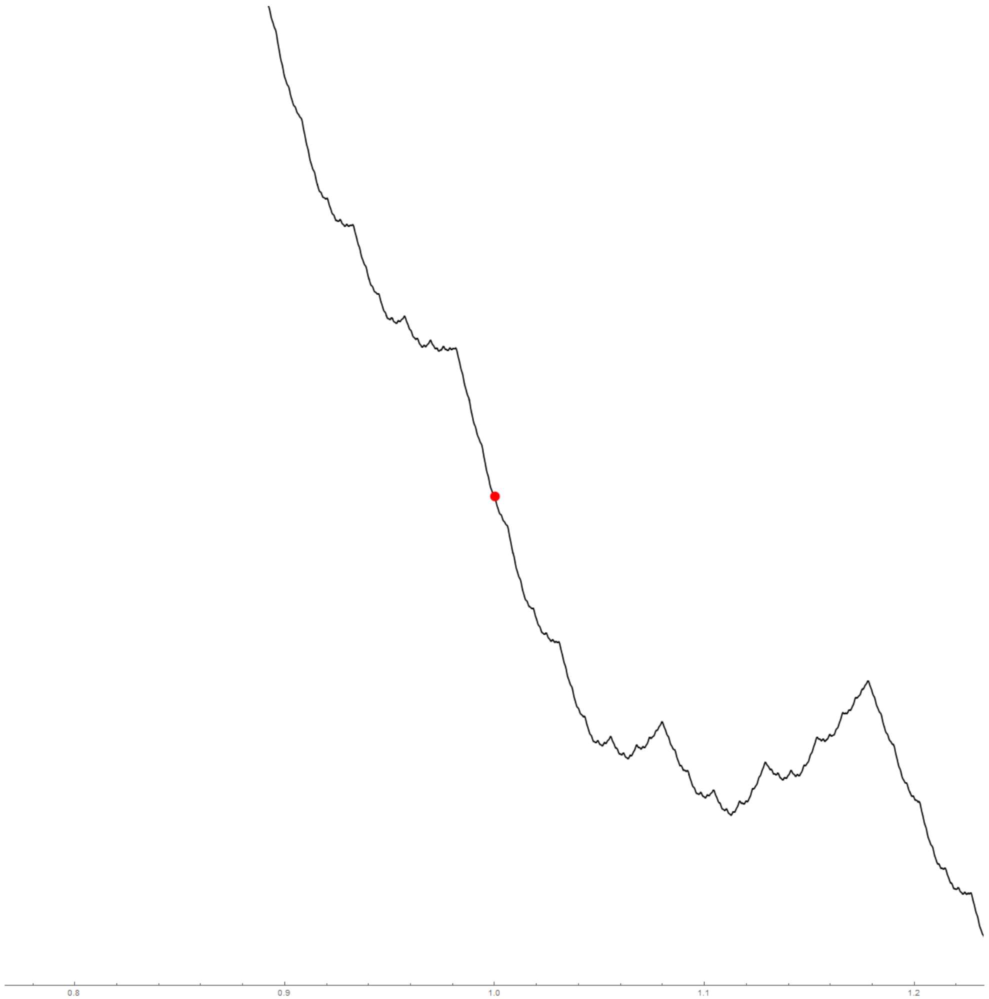

Again, no line!

The impact of Weierstrass' discovery

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Actually, far from being the exception, in a precise sense, it is **normal** for a continuous function to be nowhere differentiable! The smooth functions are the **wierd** ones!

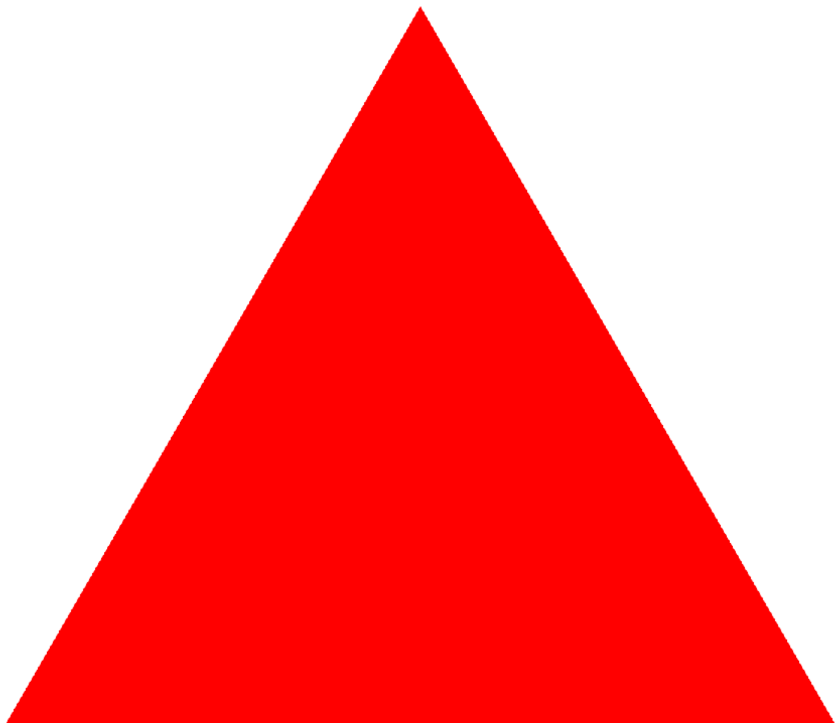
The joys of endless eating

By a process of ‘endless eating’, applied to an equilateral triangle in 2 dimensions, we obtain the so-called **Sierpiński Triangle** (1915).

Stage 1 – start with an equilateral triangle having side length 1.

Area $= \frac{1}{2}$

Boundary length $= 3$



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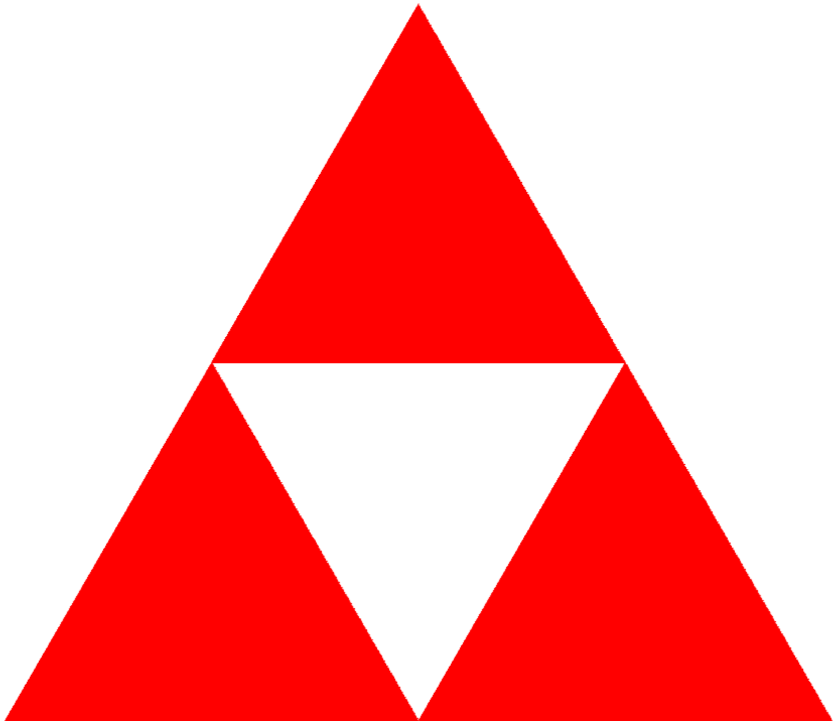
Stage 2 – split the triangle into four equal equilateral triangles having side length $\frac{1}{2}$ and ‘eat’ the middle one.

Area $= \frac{1}{2} \cdot \frac{3}{4} = 0.375$

Boundary length $= 4.5$



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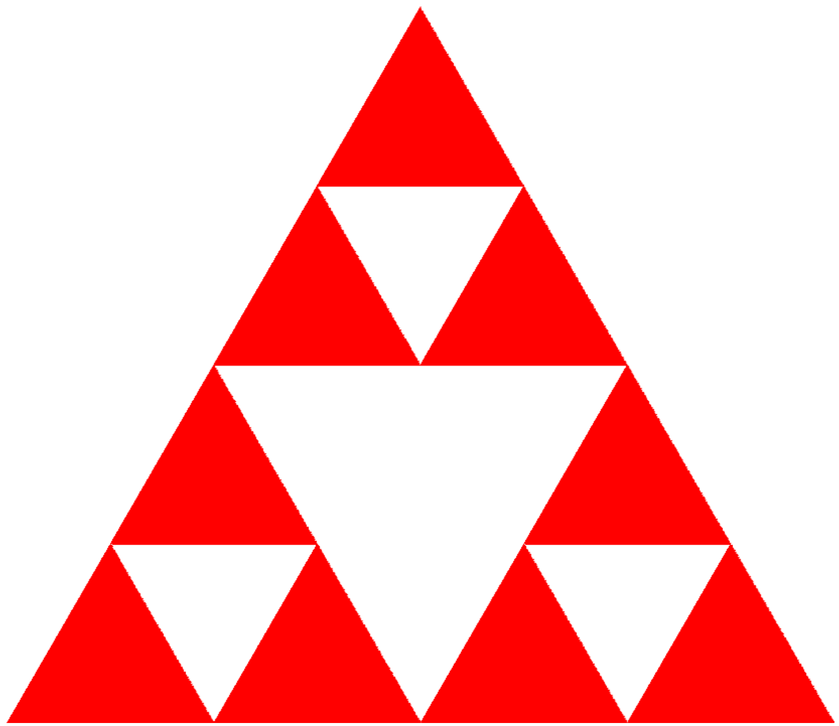
Stage 3 – we eat the middle miniature equilateral triangle from each of the remaining three triangles.

$$\text{Area} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} \approx 0.2813$$

$$\text{Boundary length} = 6.75$$



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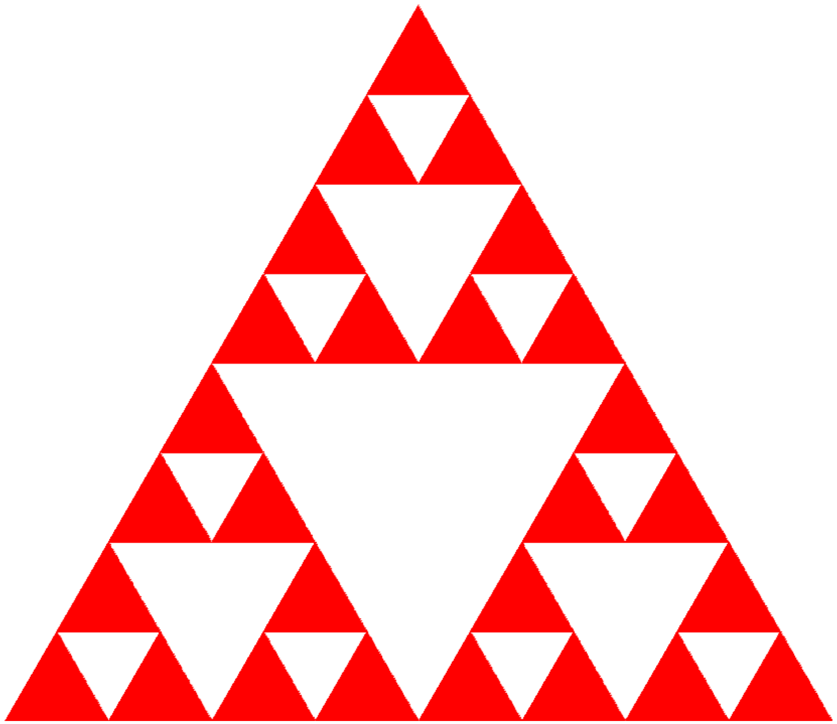
Stage 4 – we eat yet again...

$$\text{Area} = \frac{1}{2} \cdot \left(\frac{3}{4}\right)^3 \approx 0.2109$$

$$\text{Boundary length} = 10.125$$



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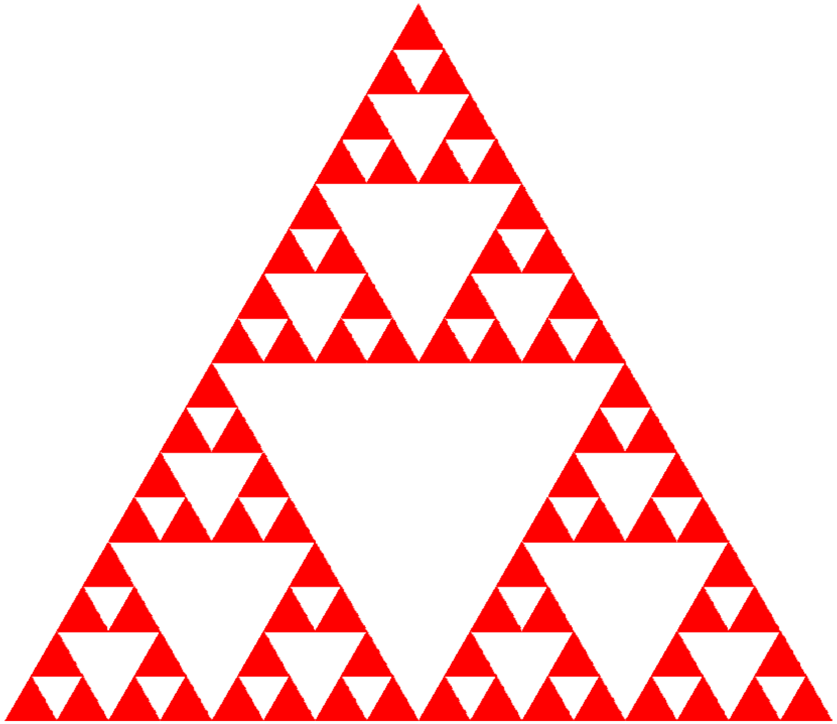
. . . and again. . .

Area $= \frac{1}{2} \cdot \left(\frac{3}{4}\right)^4 \approx 0.1582$

Boundary length ≈ 15.188



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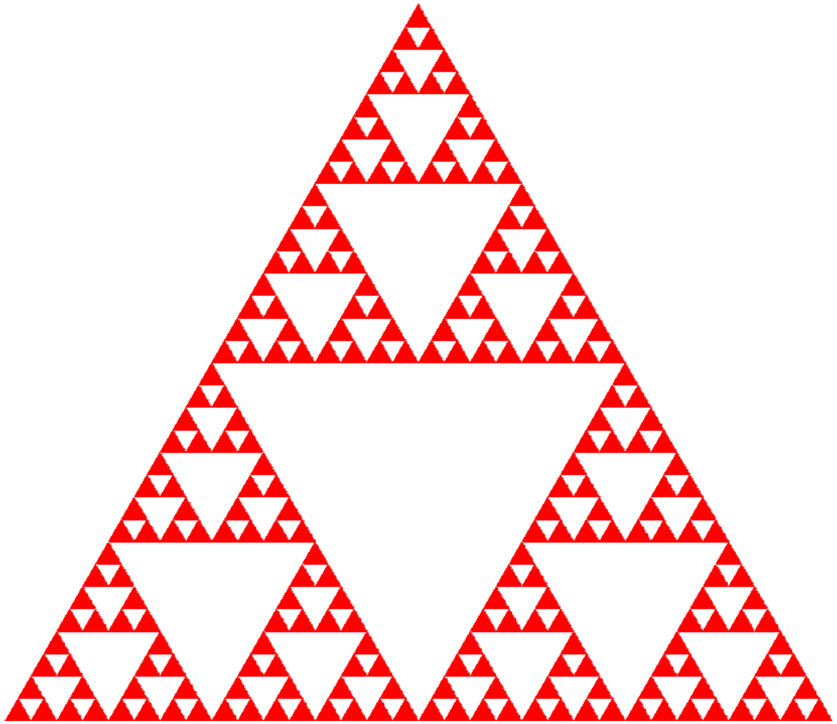
...for ever and ever...

Area $= \frac{1}{2} \cdot \left(\frac{3}{4}\right)^5 \approx 0.1187$

Boundary length ≈ 22.781



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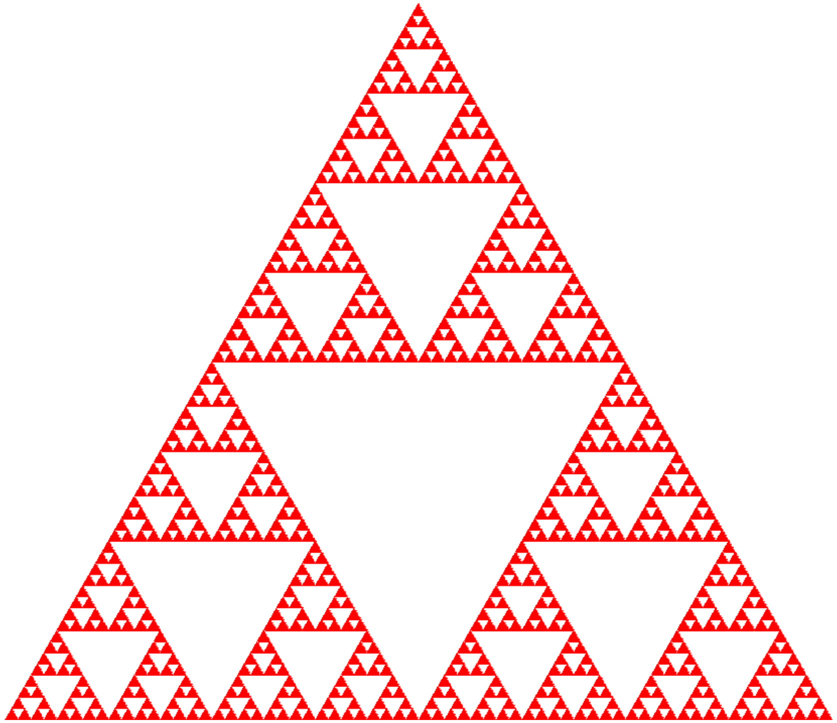
Area $= \frac{1}{2} \cdot \left(\frac{3}{4}\right)^6 \approx 0.0890$

Boundary length ≈ 34.172



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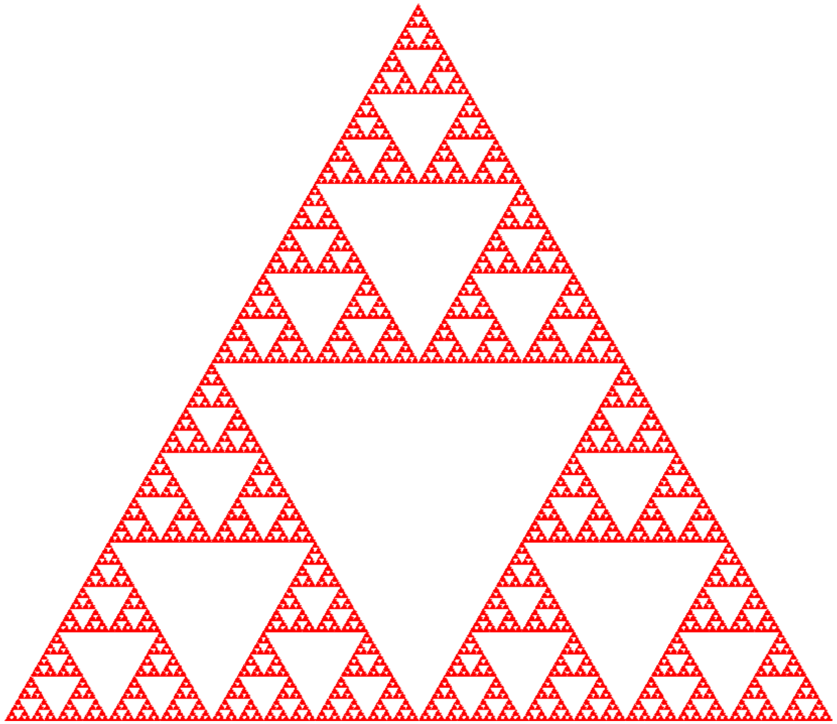
Area $= \frac{1}{2} \cdot \left(\frac{3}{4}\right)^7 \approx 0.0667$

Boundary length ≈ 51.258



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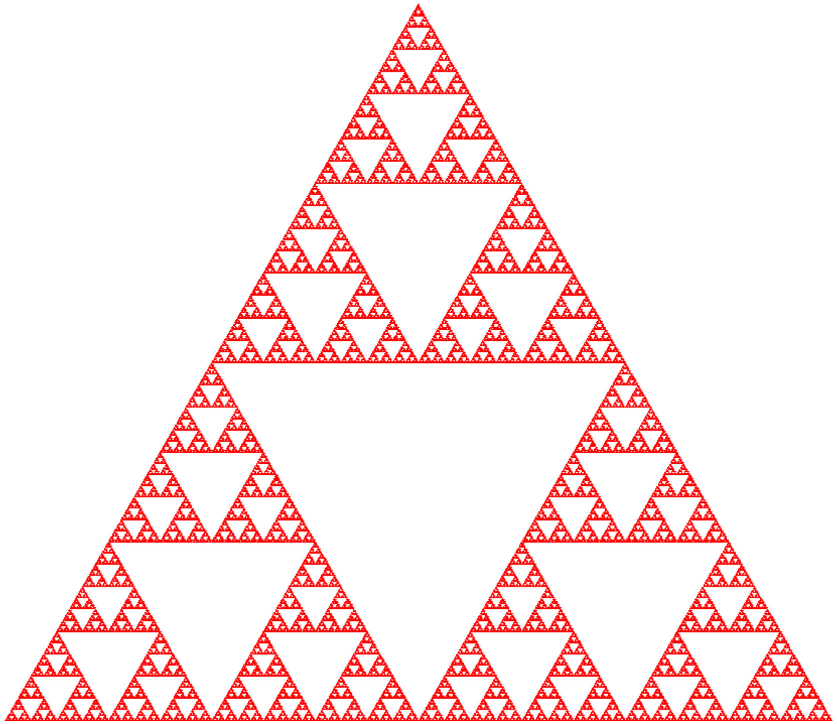
Area $= \frac{1}{2} \cdot \left(\frac{3}{4}\right)^8 \approx 0.0501$

Boundary length ≈ 76.887



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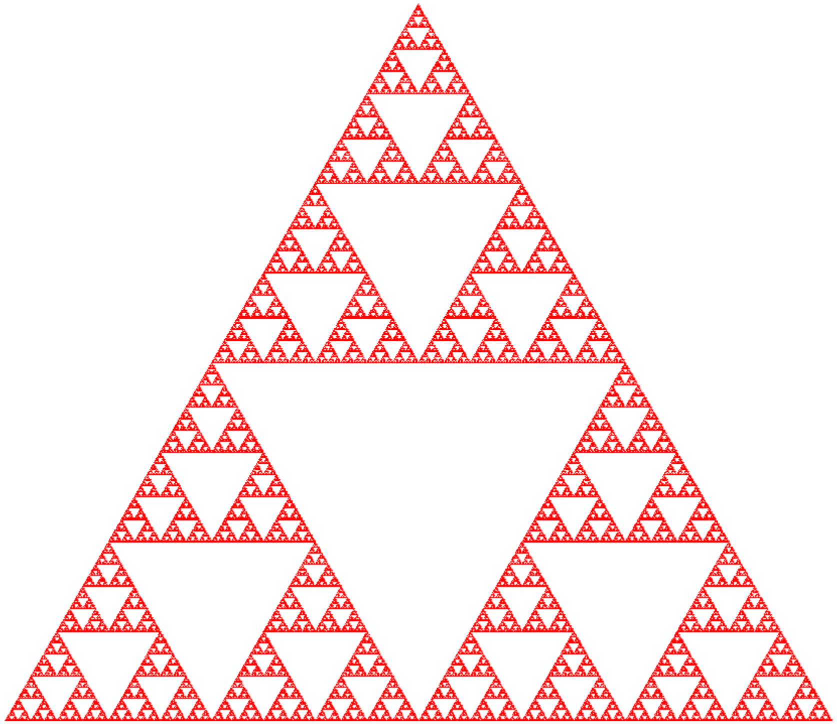
$$\text{Area} = \frac{1}{2} \cdot \left(\frac{3}{4}\right)^9 \approx 0.0375$$

$$\text{Boundary length} \approx 115.33$$



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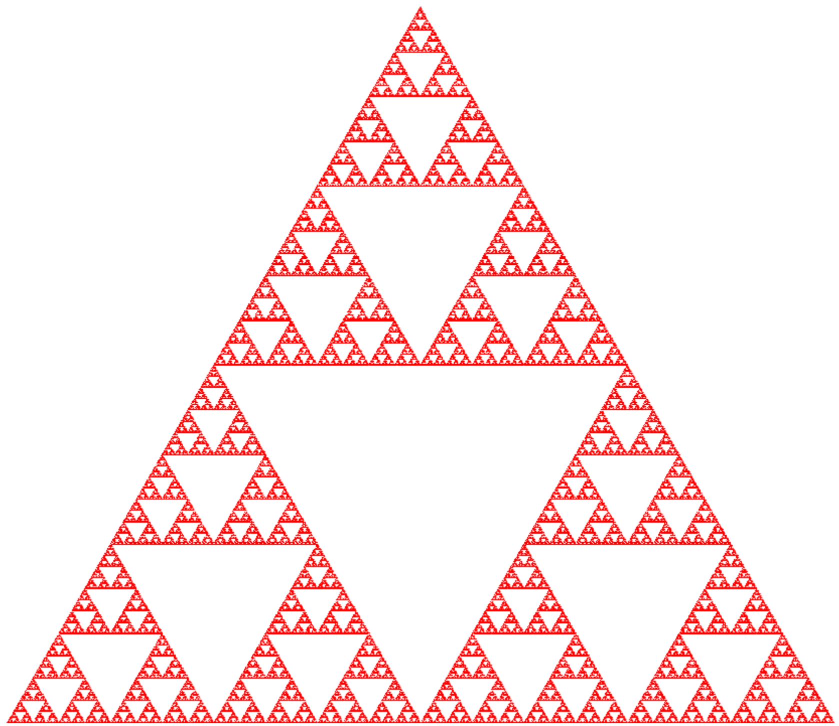
After **infinitely** many steps, we are left with an object having **zero surface area** but **infinite boundary length**. It is also **self-similar**.

Area $= 0$

Boundary length $= \infty$



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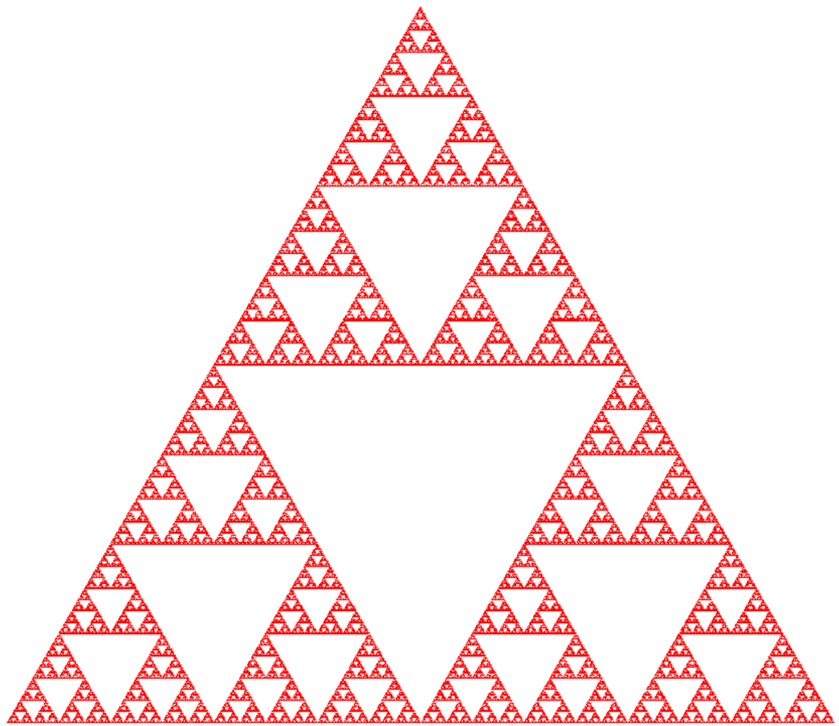
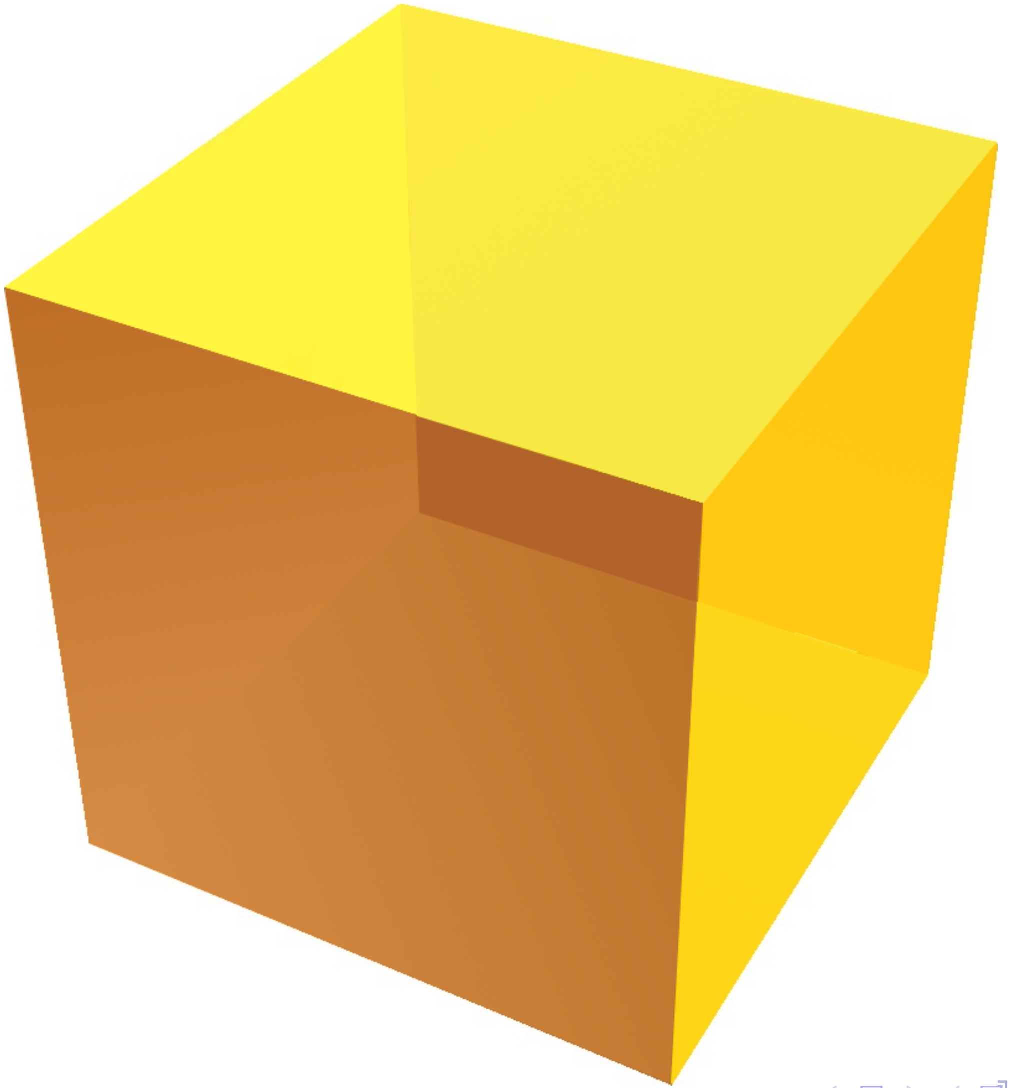


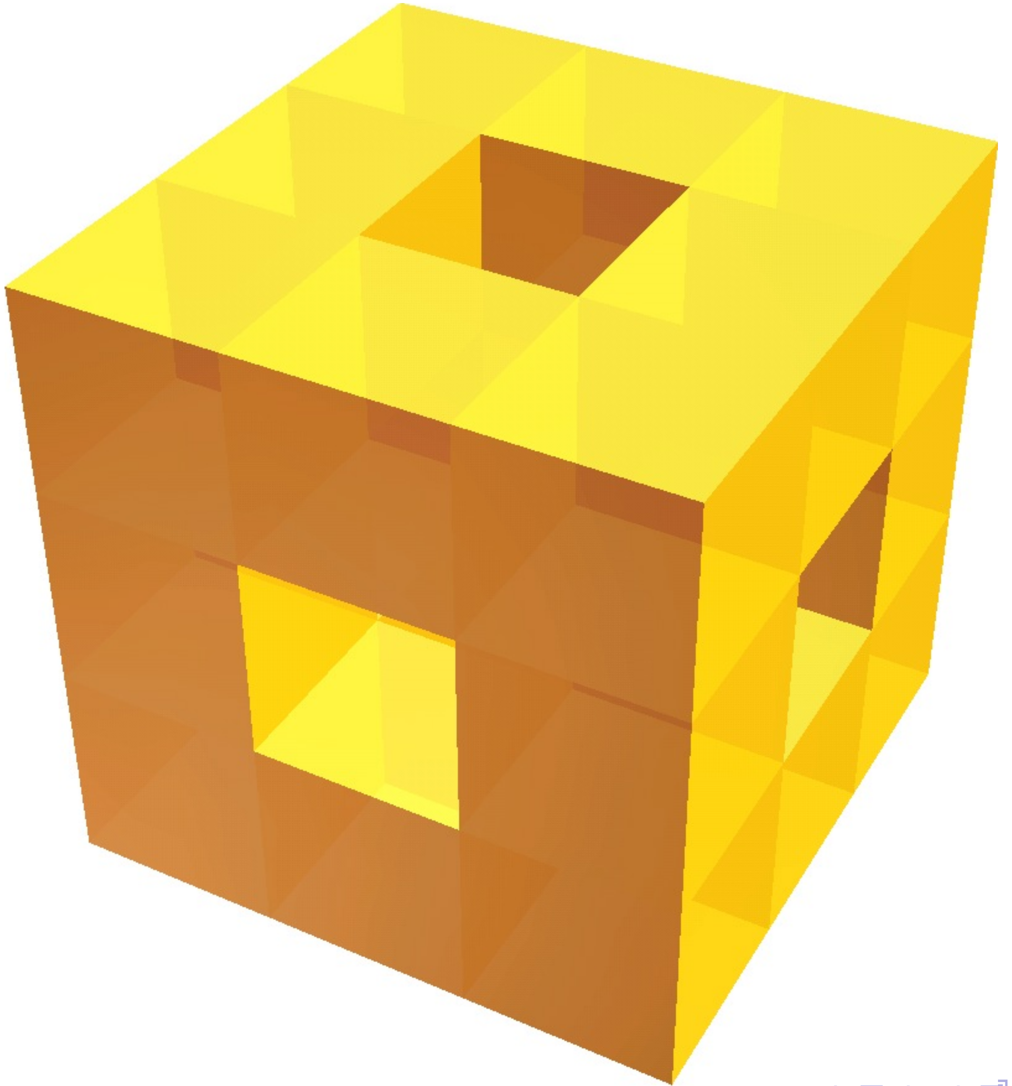
Another fractal obtained by eating

Stage 1 – begin with a cube.



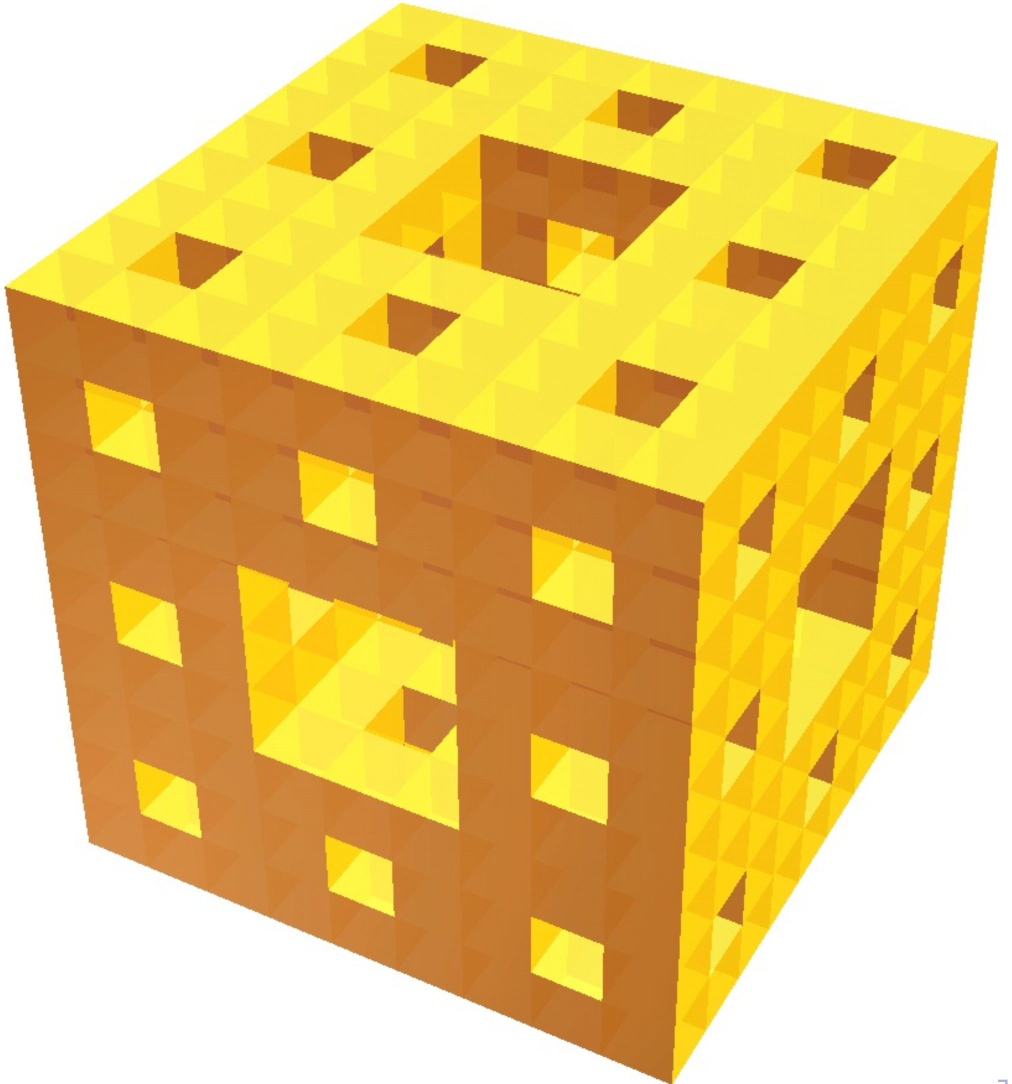
Another fractal obtained by eating

Stage 2 – split the cube into 27 sub-cubes, remove the sub-cubes in the middle of each face, and the cental sub-cube, leaving 20 sub-cubes.



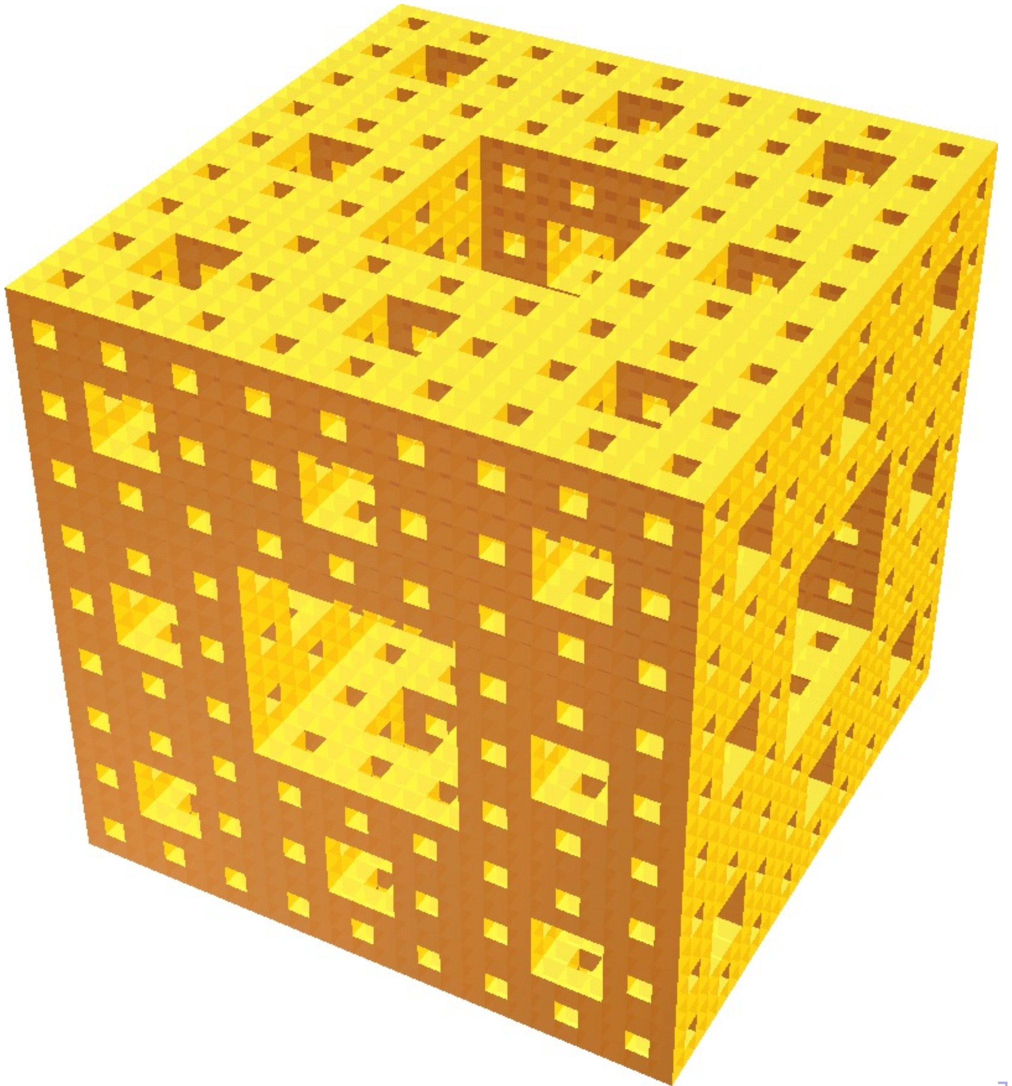
Another fractal obtained by eating

Stage 3 – repeat the above process with each of the remaining 20 sub-cubes, leaving $20 \times 20 = 400$ even smaller cubes.



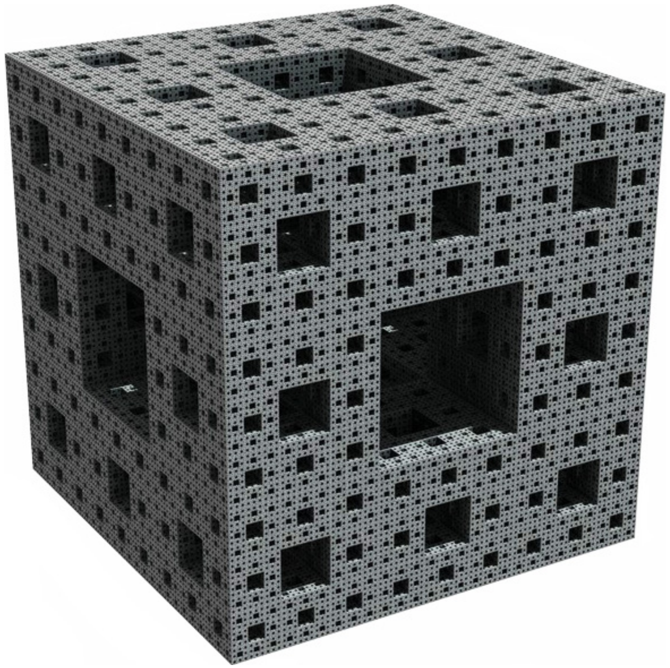
Another fractal obtained by eating

Stage 4 – repeat with the remaining 400 smaller cubes, to give $400 \times 20 = 8000$ even smaller cubes. . . , and so it continues . . .



Another fractal obtained by eating

The **Menger Sponge** (1926) is the object that you have left after this process has been repeated **infinitely many** times.



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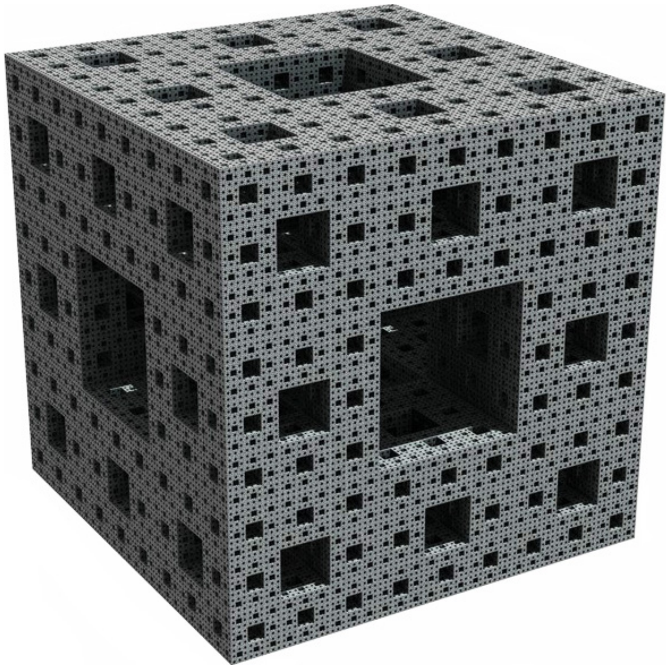

It is an object having **zero volume**, but **infinite surface area**. . . .

Iterated function systems

Rather than eating endlessly, we generate fractals using so-called **iterated function systems** (to iterate is to repeat something over and over).

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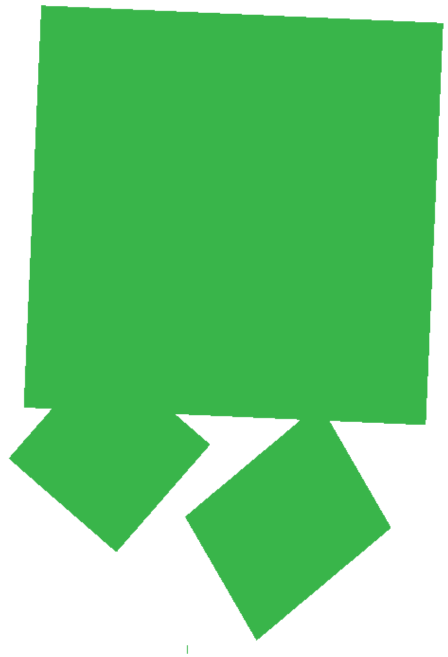

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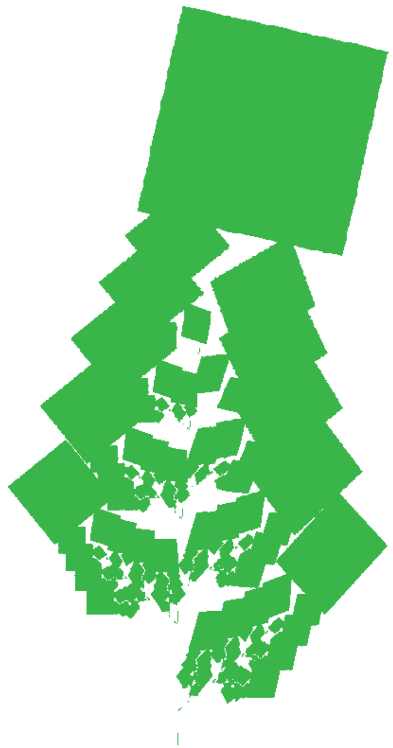

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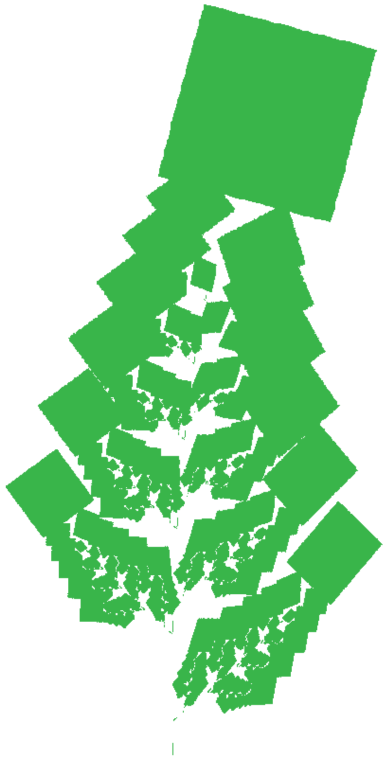

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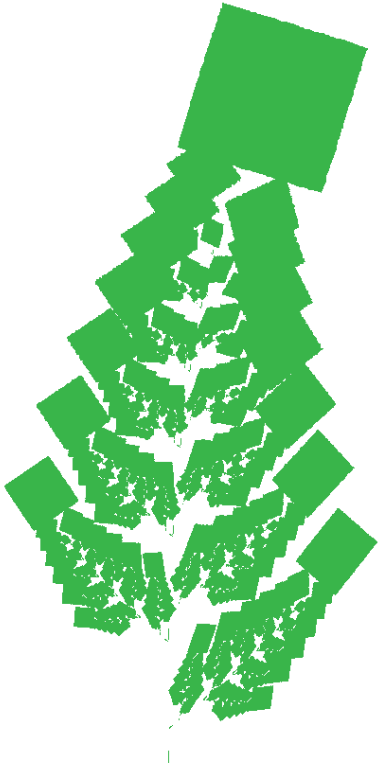

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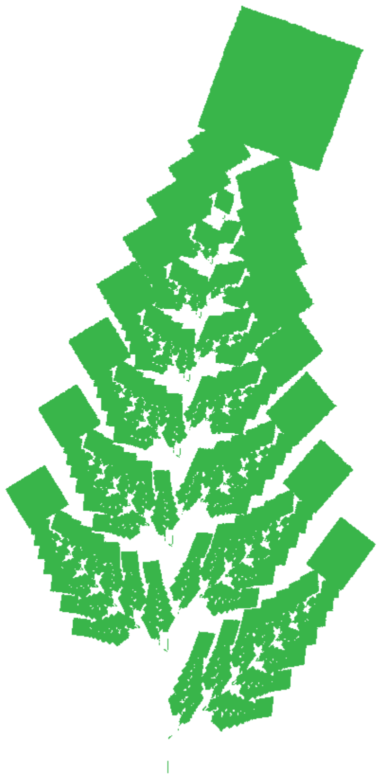

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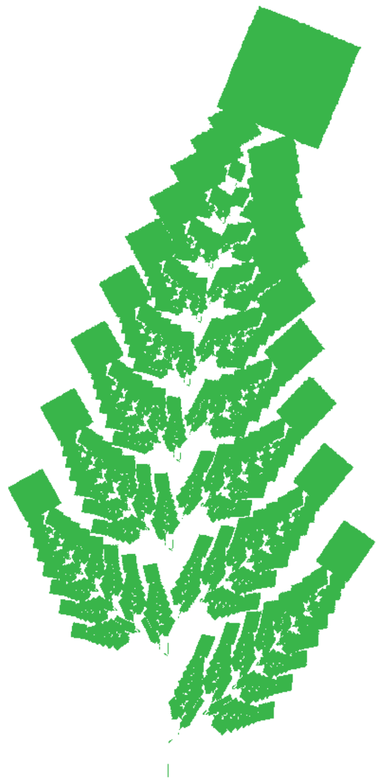

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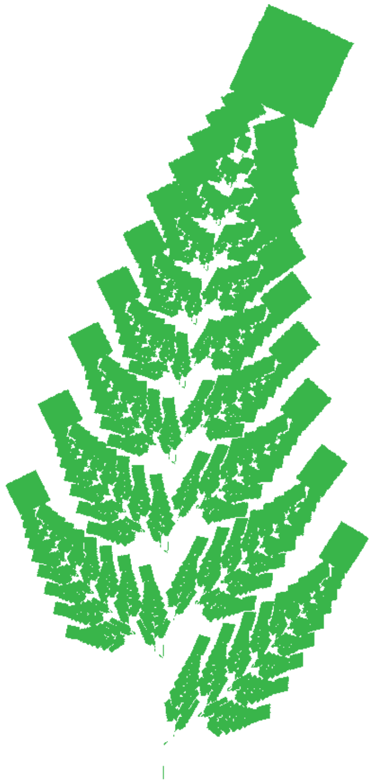

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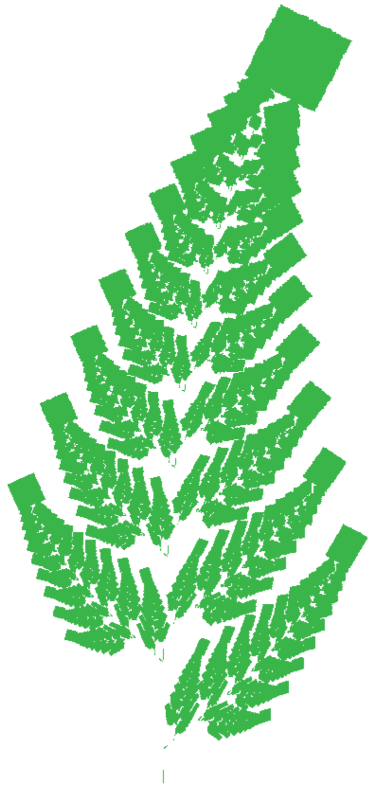

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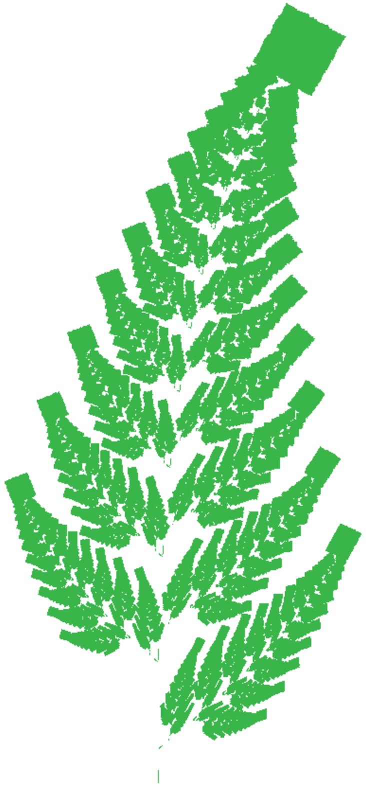

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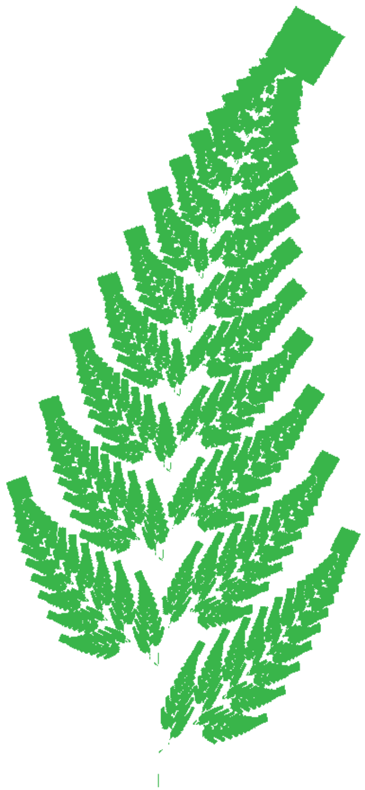

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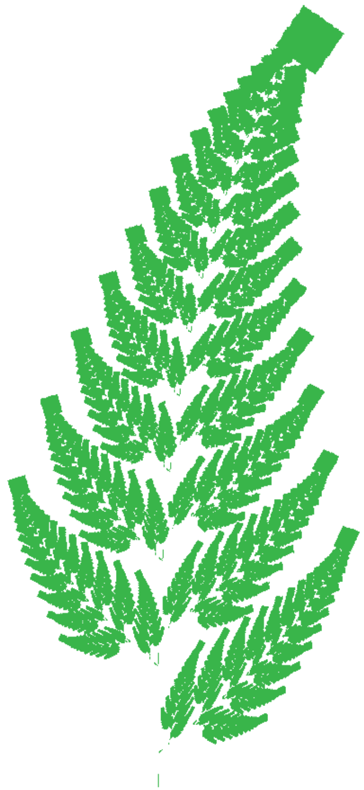

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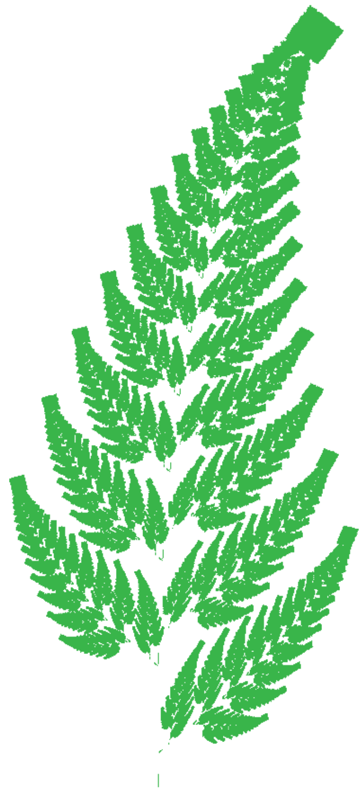

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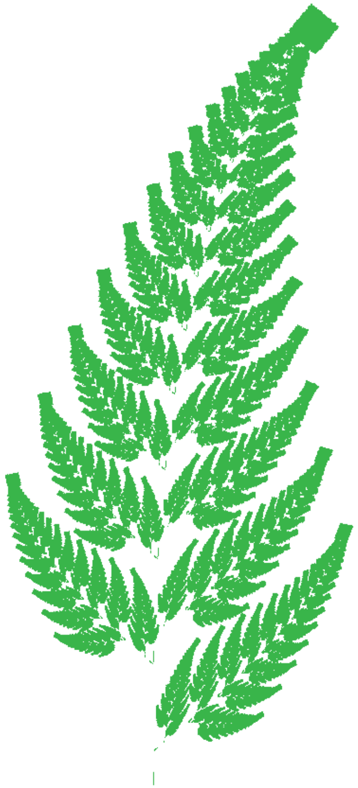

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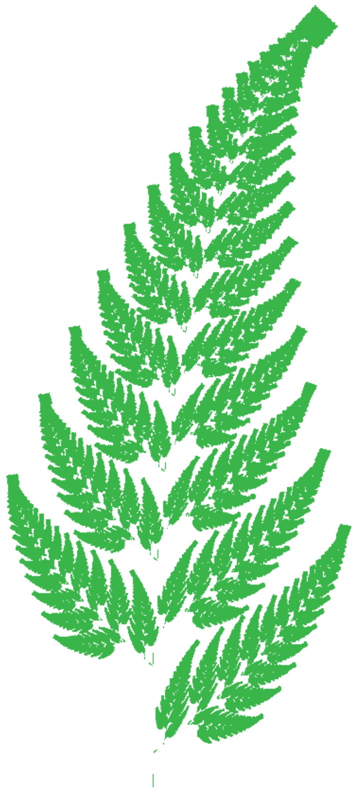

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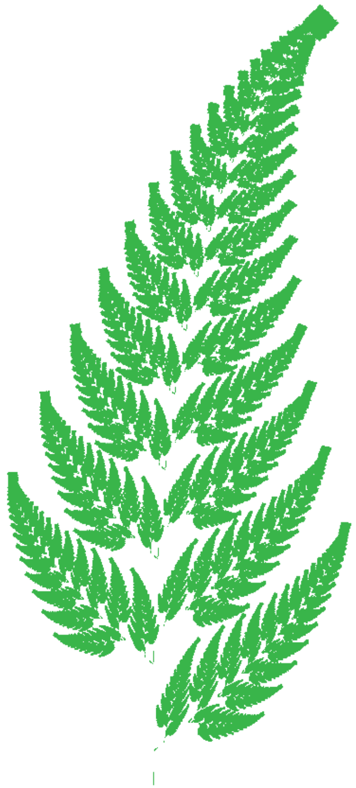

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After infinitely many steps we get the so-called **Barnsley Fern**.

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After infinitely many steps we get the so-called **Barnsley Fern**.

It is generated by a system of just four simple geometric transformations.

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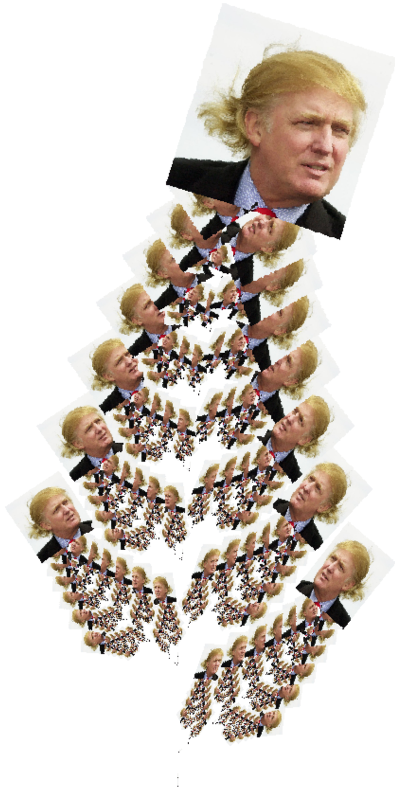

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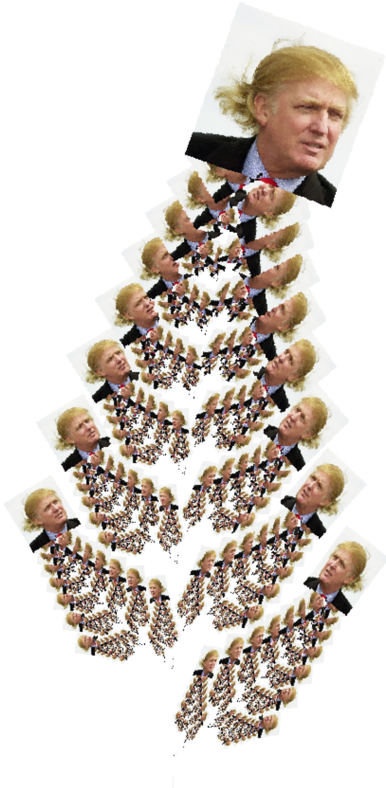

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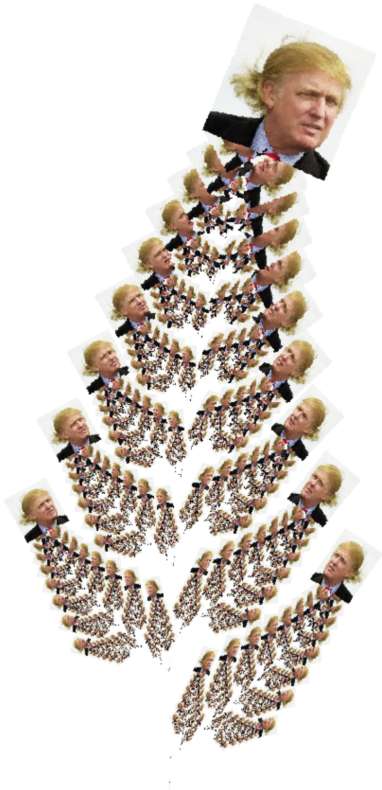

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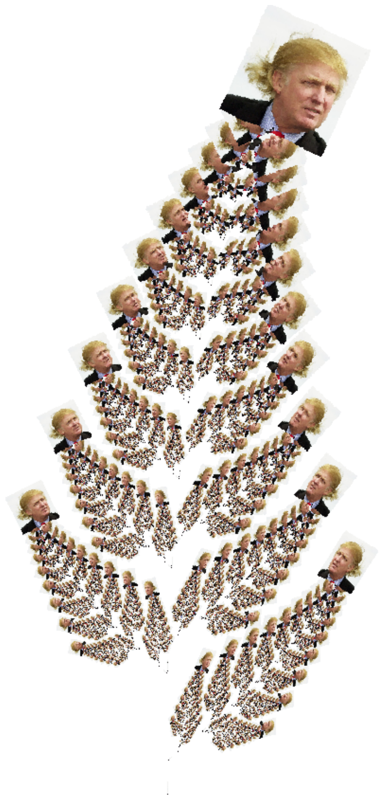

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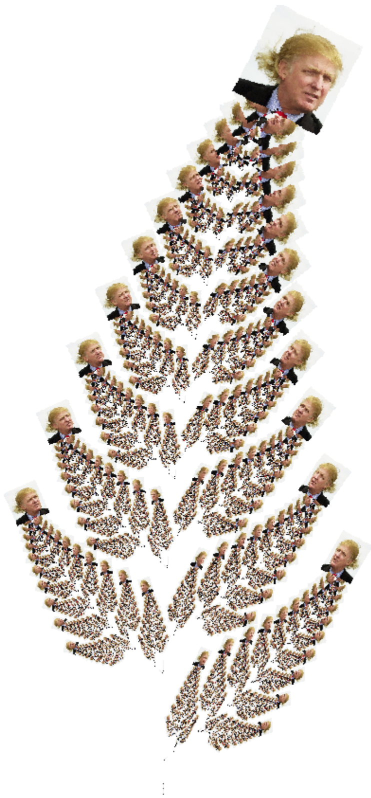

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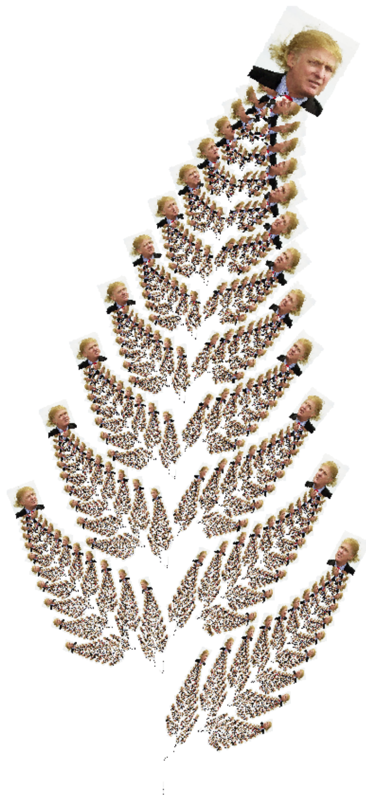

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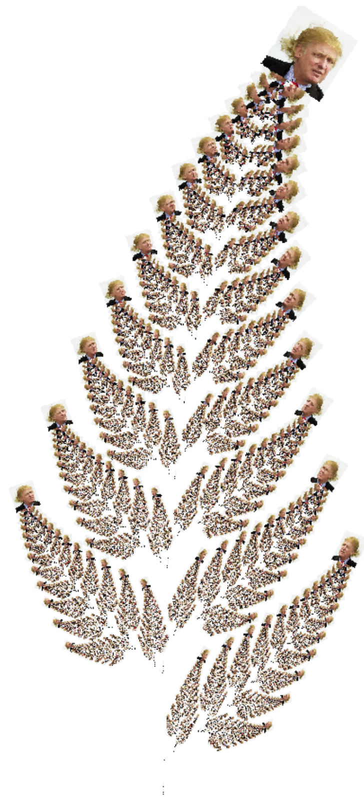

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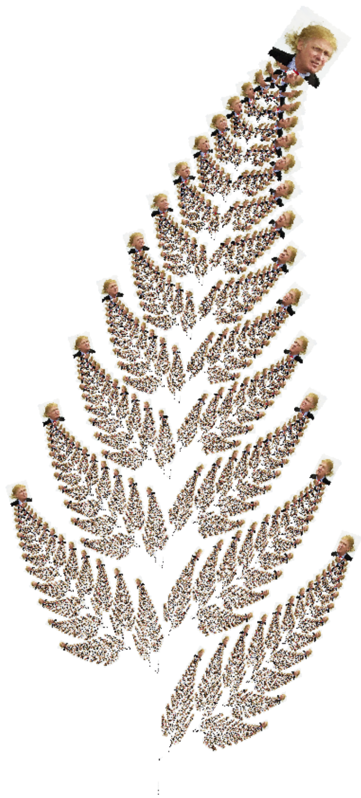

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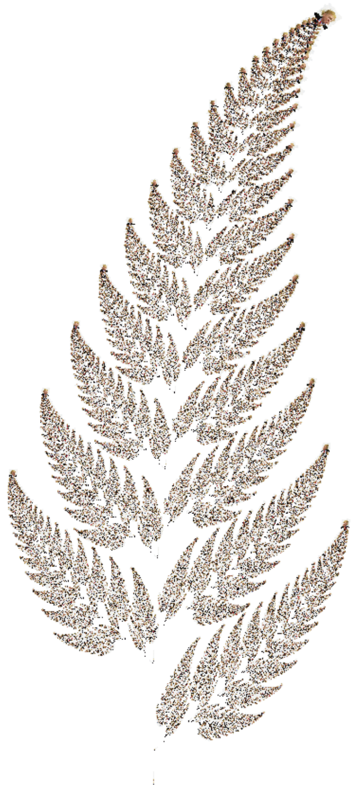

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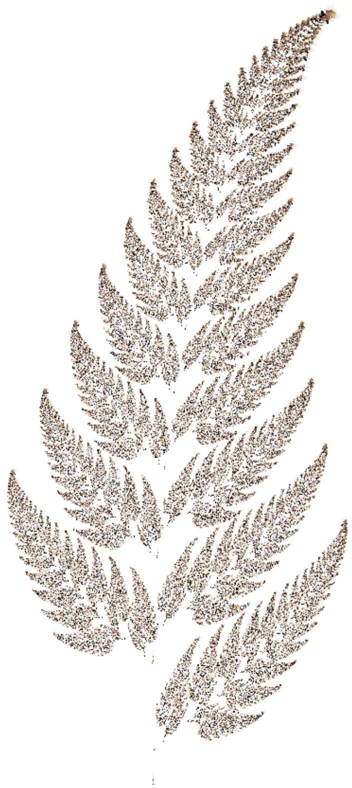

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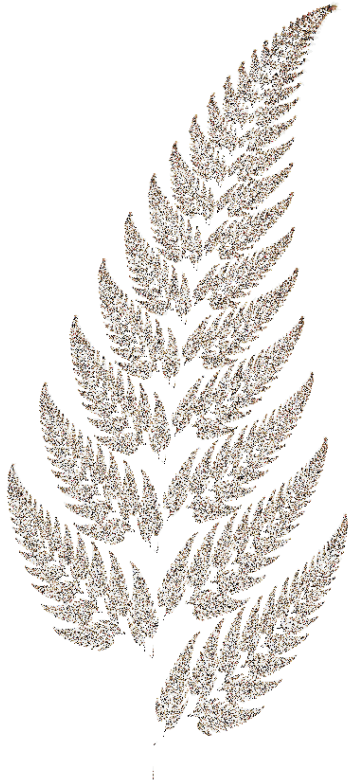

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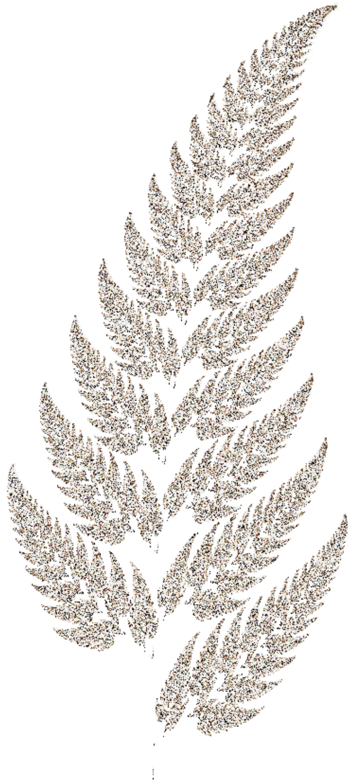

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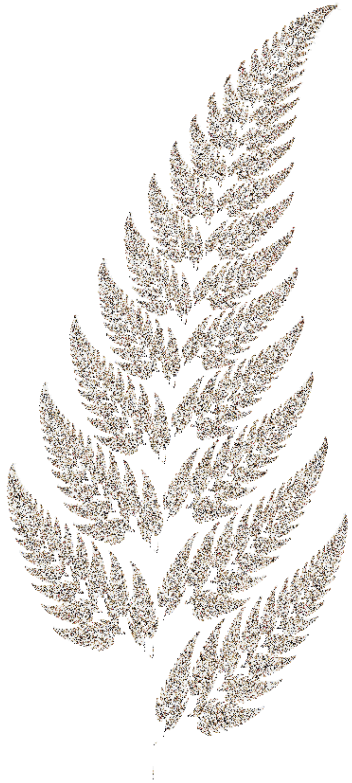

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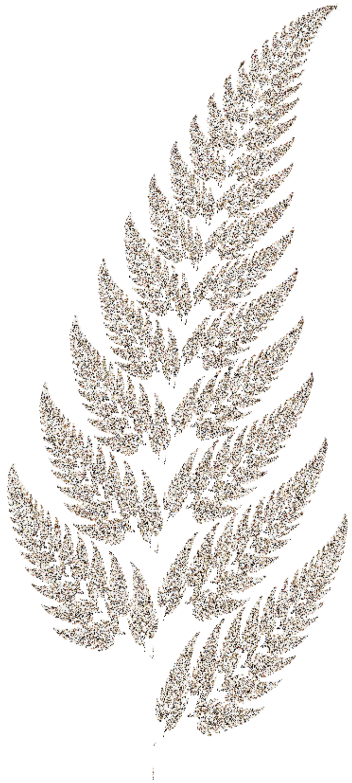

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Whether you start with a green block or a mendacious Twitter troll, you end up with the same shape!

What do these monsters have to do with nature?

It turns out that many natural structures are fractal-like – and with good reason.

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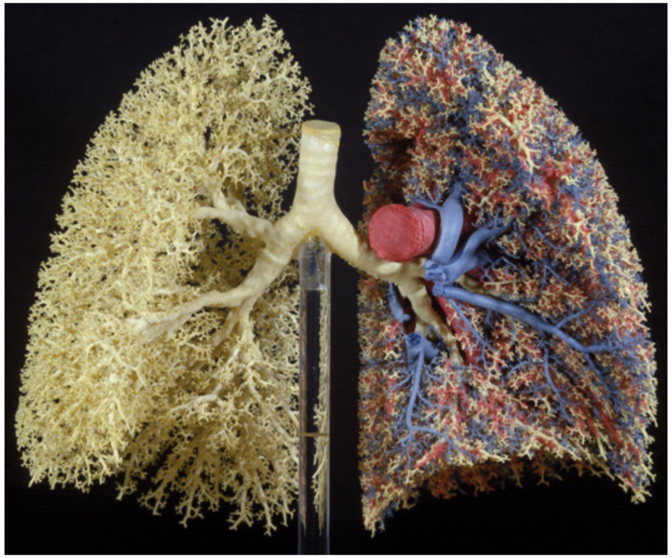
Take the airways in human lungs.



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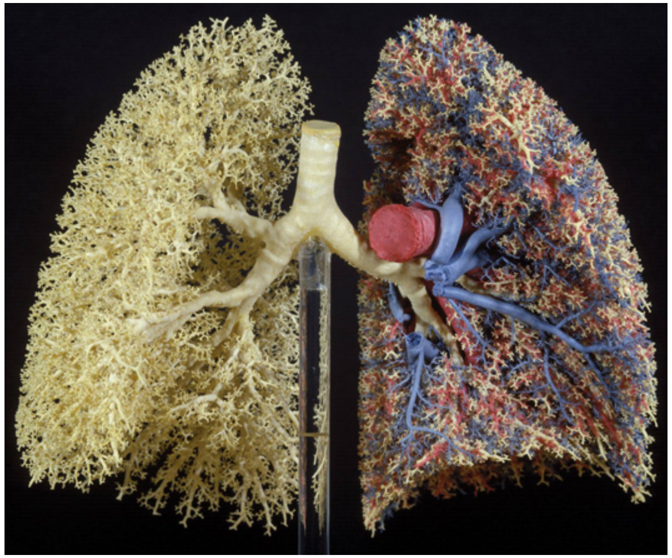

This structure has huge surface area, despite small volume. . .

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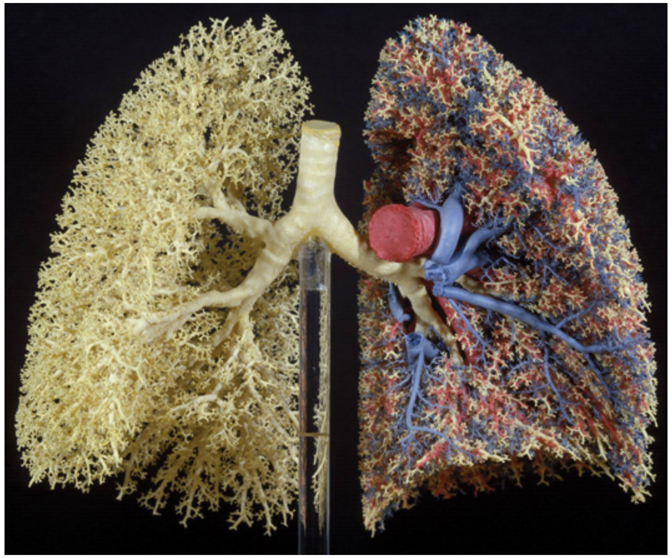

This structure has huge surface area, despite small volume...

...like the Menger sponge.

Self-similarity in nature

Romanesco broccoli exhibits self-similarity.

Self-similarity in nature

As do ferns. . .

Self-similarity in nature

. . . and snowflakes. . .

Self-similarity in nature

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The apparent connection between nature and fractals was first popularised by Benoît Mandelbrot in his book ‘The Fractal Geometry of Nature’.

Are we in chaos?

In 1963, **Edward Lorenz** created a simple system of **differential equations** to model atmospheric convection:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y, \quad \text{and} \quad \frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z,$$

where x corresponds to convection rate and y and z temperature variation.

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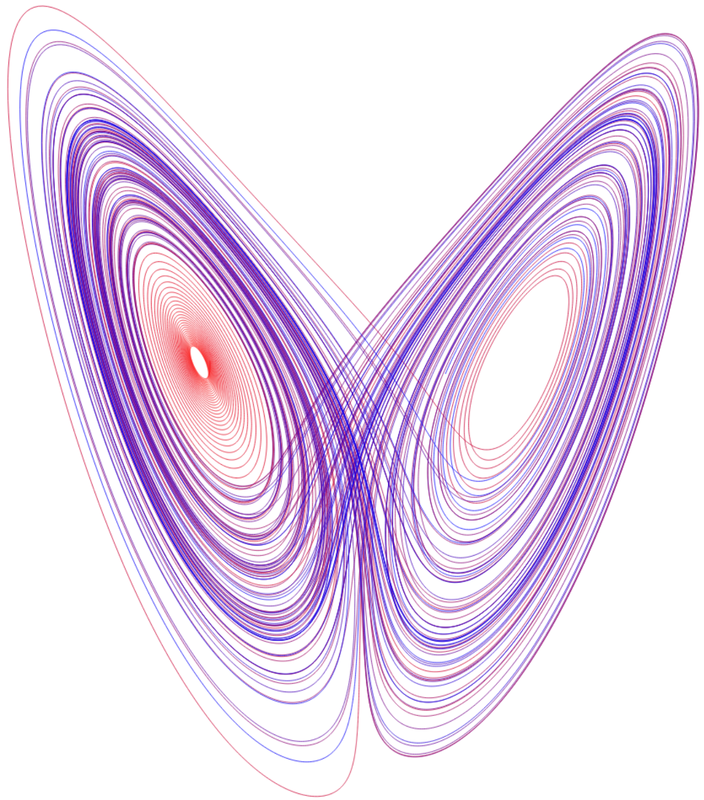
Many systems of equations that model physical processes are chaotic. Even though such systems are **deterministic** (roughly, non-random), this can make solving them very difficult.

Fractal attraction

While chaotic systems are very hard to solve, sometimes order can be imposed. The following is an example of a **Lorenz attractor**.



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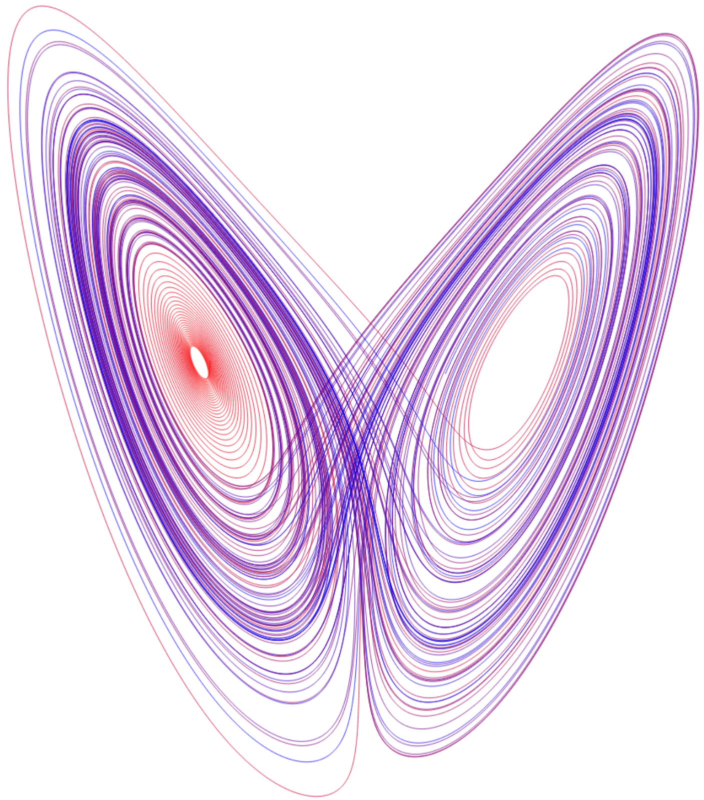

If two particles start close together in the attractor, then while their orbits may diverge wildly over time, **both orbits will stay within the attractor**.

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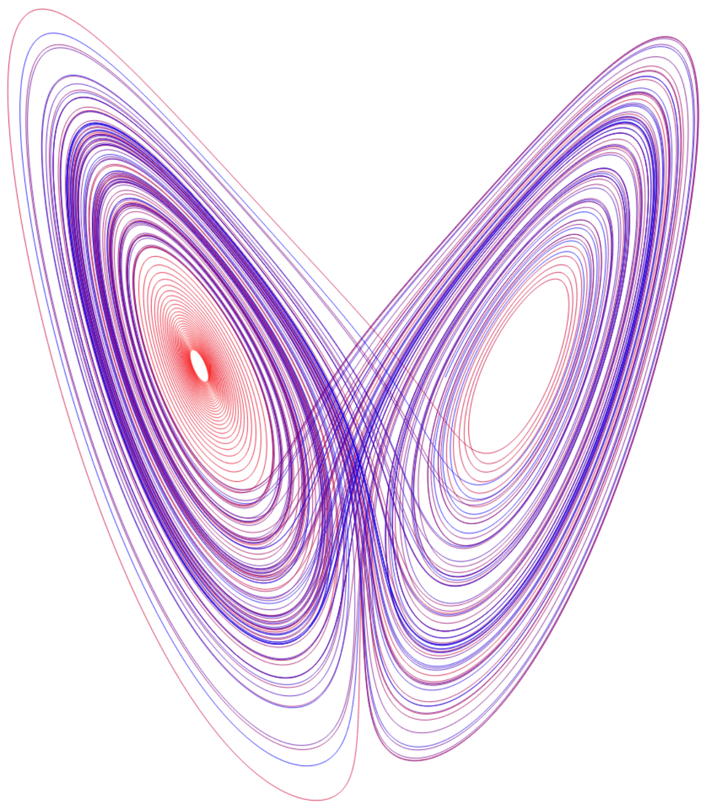

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The cross section of this attractor is a fractal.

Fractals in physics, finance and technology

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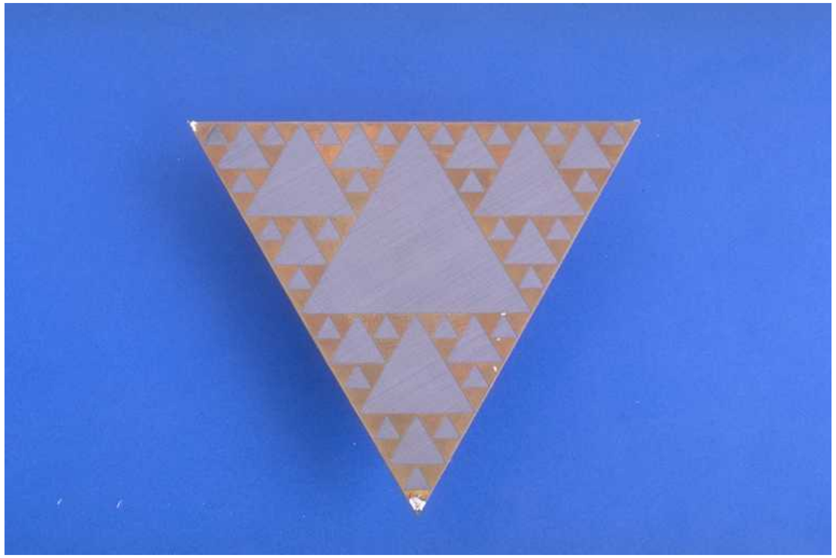
The designs of so-called **fractal antennae** are inspired by fractal geometry. This is the so-called **Sierpiński antenna**.



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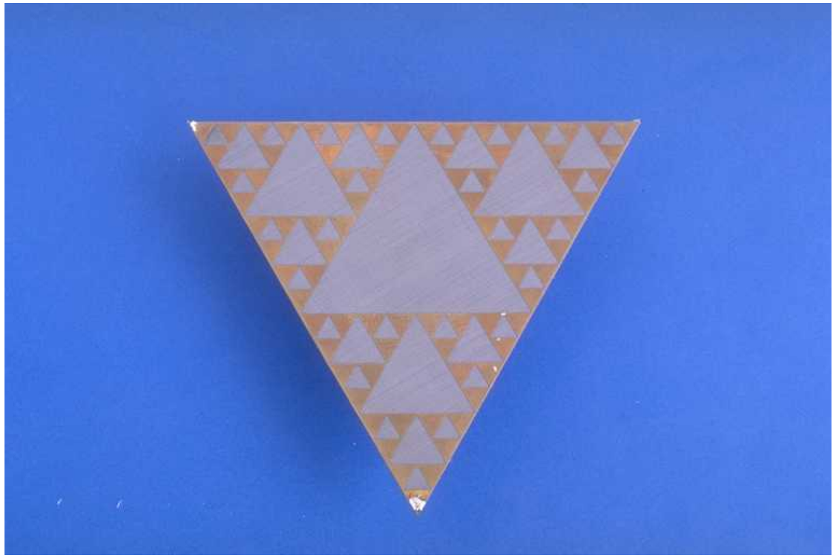

Given the usefulness of fractal-like structures in nature, it would not surprise me to see technological applications of fractals in the future.

Another dimension to fractals

Points, straight lines, discs and spheres have dimension 0,1,2 and 3, respectively. Einstein's 'space-time' is 4-dimensional.

Another dimension to fractals

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Following this intuition we arrive at the notion of **Hausdorff dimension**, which can assign dimensions to very complicated objects, including fractals. Often these dimensions are **not** integers.

Another dimension to fractals

Object	Hausdorff dimension
Point	0
Straight line	1
Disc	2
Sphere	3
Sierpiński Triangle	$\log 3/\log 2 \approx 1.585$
Menger Sponge	$\log 20/\log 3 \approx 2.727$
Weierstrass Functions	Between 1 and 2 (unknown in general)

Another dimension to fractals

Given a metric space (X, d), the Hausdorff dimension of X is defined to be

$$\lim_{n\to\infty} H_n^d(X),$$

where

$$H_n^d(X) := \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam}(E_i)) : \bigcup_{i=1}^{\infty} E_i = X \text{ and } \operatorname{diam}(E_i) < n^{-1}\right\},$$

and

$$\operatorname{diam}(E) := \sup\{d(x,y) : x, y \in E\}.$$

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  `https://commons.wikimedia.org/wiki/File:Sierpinski_zoom_2.gif`
- **tthsqe12** – Mandelbrot zoom video.

  `https://www.youtube.com/watch?v=PD2XgQOyCCk`
- **Lord8Vader** – Lorenz attractor video.

  `https://www.youtube.com/watch?v=dP3qAq9RNLg`
- **Wikipedia** – Various images.
- **imgflip.com** – Angry cat.